Statistical Issues in Designing an Optimal Detection System with Multiple Heterogeneous Sensors

Carol Y. Lin
Lance A. Waller
Robert H. Lyles
P. Barry Ryan

Outline

- Introduction
- Expected cost and Bayesian optimal decision criterion with minimum cost to select an optimal system design
- Prob. of detection and false alarm rate of a system with independent and correlated heterogeneous individual tests
- Impact of system size and correlations on the system performance
- Example

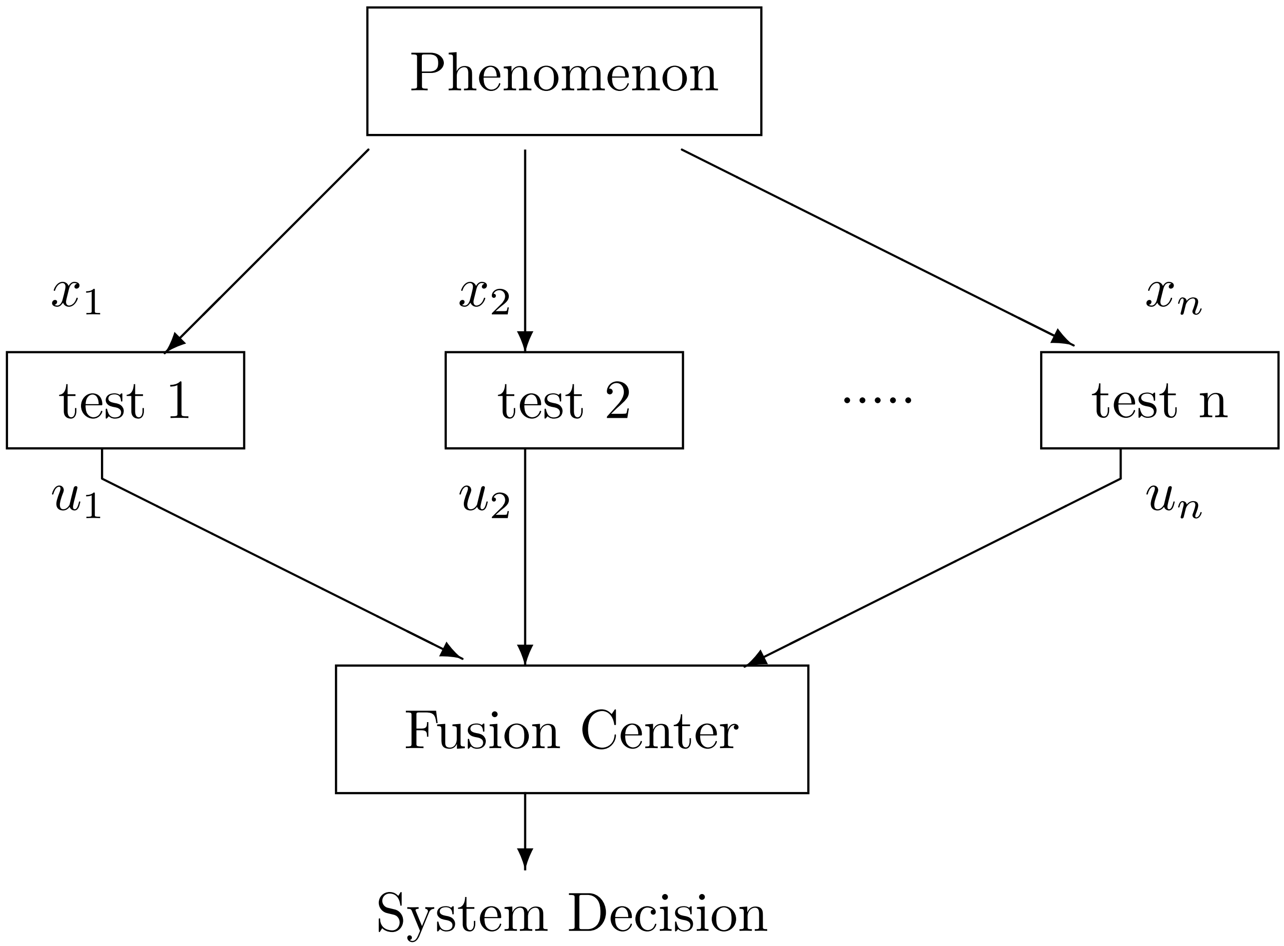
Purpose of Detection

- Provide reliable information on the presence or absence of underlying phenomenon (disease status, critical levels of air pollution) to allow early intervention
  - EPA guideline: Air Quality Index above 100
  - WHO guideline:
    - PM_{10} short-term concentration exceeding $50ug/m^3$
    - NO_2 short-term concentrations exceeding $200ug/m^3$

Optimal System Design

- Rationale: Combination of multiple (homogeneous or heterogeneous) tests improves diagnostic accuracy
- Cost Consideration:
  - Costs of (heterogenous) tests
  - System budget constraint
  - Costs of false alarm and false negative
- Interest: design an optimal detection system from a set of homogeneous or heterogeneous sensors subject to the monetary constraint

Detection System



Example

- Underlying phenomenon: the level of nitrogen dioxide concentration exceeds some pre-specified level
- Detection system: an air pollution monitor network to detect pollution caused by airport traffic in Logan International Airport and surrounded community
- Sampling sites: end of runways, runway intersection, busy roadways in the airport and surrounded community

Notation

- H_0: phenomenon absent and H_1: phenomenon present
- $P(H_0)$ and $P(H_1)$: *a priori* probabilities of the two hypothesis
- x_i: an observation from i^{th} test
- u_i: the individual binary decision and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ a vector of test decisions
- U_f: a combined decision at the fusion center (0,1)
- P_{D_i}: probability of detection for the i^{th} individual test
- P_{F_i}: false alarm rate for the i^{th} individual test
- P_D: probability of detection
- P_F: false alarm rate
- $C_{r,s}$: the cost of deciding H_r when H_s is true. For example, C_{10}: cost of false alarm

Assumptions

- Probability of detection and false alarm of individual tests (P_{D_i} and P_{F_i}) estimated
- Prior probability that phenomenon is presented $[P(H_1)]$ estimated
- Costs of all decisions ($C_{r,s}$) estimated

Decision Criteria

We need the following:

- Criterion to select an optimal system
  - Minimum risk (expected cost)
- Criterion to combine decisions from individual tests (sensor)
  - Bayesian optimal decision criterion with minimum cost

Expected Cost

$$
\begin{aligned}
R &= E(C) = \sum_{r=0}^{1}\sum_{s=0}^{1} C_{r,s} P(U_f = r, H_s) \\
&= \sum_{r=0}^{1}\sum_{s=0}^{1} C_{r,s} P(U_f = r | H_s) P(H_s)
\end{aligned}
$$

$$
R^A \underset{A}{\overset{B}{\gtrless}} R^B
$$

$$
R^A - R^B = P(H_1)(C_{01} - C_{11})[P_D^B - P_D^A] + P(H_0)(C_{10} - C_{00})[P_F^A - P_F^B]
$$

$$
P_D = P(U_f = 1 | H_1) \text{ and } P_F = P(U_f = 0 | H_0)
$$

Bayesian Optimal Decision Criterion with Minimum Cost

$$\Lambda(\mathbf{u}) = \frac{P(\mathbf{u}|H_1)}{P(\mathbf{u}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})} = \gamma$$

(Chair and Varshney, 1986; Aalo and Viswanathan, 1989)

$$\Longleftrightarrow \frac{P(\mathbf{u}|H_1)}{P(\mathbf{u}|H_0)} \frac{P(H_1)}{P(H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})}{(C_{01} - C_{11})}$$

$$\Longleftrightarrow \frac{P(H_1|\mathbf{u})}{P(H_0|\mathbf{u})} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})}{(C_{01} - C_{11})}$$

$$\Longleftrightarrow E(C|\mathbf{u}, U_f = 0) \underset{H_0}{\overset{H_1}{\gtrless}} E(C|\mathbf{u}, U_f = 1)$$

.

Likelihood Ratio of Conditionally Independent Tests

$$P(u_i|H_1) = (P_{D_i})^{U_i}(1 - P_{D_i})^{1-U_i}$$

$$P(u_i|H_0) = (P_{F_i})^{U_i}(1 - P_{F_i})^{1-U_i}$$

$$\Lambda(\mathbf{u}) = \frac{P(\mathbf{u}|H_1)}{P(\mathbf{u}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})} = \gamma$$

$$\Longleftrightarrow \prod_{i=1}^{n} \left(\frac{P_{D_i}}{P_{F_i}}\right)^{U_i} \left(\frac{1 - P_{D_i}}{1 - P_{F_i}}\right)^{1-U_i} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})} = \gamma$$

Combined Probability of Detection and False Alarm Rate

$$
\begin{aligned}
P_D &= P(U_f = 1|H_1) \\
&= P(\Lambda(\mathbf{u}) \geq \gamma|H_1) \\
&= \sum_{\Lambda(\mathbf{u})\geq\gamma} P(\mathbf{u}|H_1) \\
&= \sum_{(m_1,m_2,\ldots,m_z)\geq(k_1,k_2,\ldots,k_z)} \prod_{j=1}^{z} \binom{n_j}{m_j} (P_{D_j})^{m_j} (1 - P_{D_j})^{n_j - m_j}
\end{aligned}
$$

$$
\begin{aligned}
P_F &= P(U_f = 1|H_0) \\
&= P(\Lambda(\mathbf{u}) \geq \gamma|H_0) \\
&= \sum_{\Lambda(\mathbf{u})\geq\gamma} P(\mathbf{u}|H_0) \\
&= \sum_{(m_1,m_2,\ldots,m_z)\geq(k_1,k_2,\ldots,k_z)} \prod_{j=1}^{z} \binom{n_j}{m_j} (P_{F_j})^{m_j} (1 - P_{F_j})^{n_j - m_j}
\end{aligned}
$$

Correlated Tests with Gaussian Noise

- $x_i = a * \mu_i + \epsilon_i$, $\epsilon_i \sim N(\mathbf{0}, \mathbf{\Sigma})$
- x_i under $H_0 \sim N(\mathbf{0}, \mathbf{\Sigma})$, and under $H_1 \sim N(\mu, \mathbf{\Sigma})$, $\mu = (\mu_{\mathbf{1}}, \mu_{\mathbf{2}}, \cdots, \mu_{\mathbf{j}}, \cdots, \mu_{\mathbf{z}})$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2 & \rho & \rho & \cdots & \rho \\ \rho & \sigma^2 & \rho & \cdots & \rho \\ \rho & \rho & \sigma^2 & \cdots & \rho \\ & & & \ddots & \\ \rho & \rho & \rho & \cdots & \sigma^2 \end{pmatrix}, 0 < \rho < 1$$

- Assume decision thresholds for all tests are equal to t

Likelihood Ratio of Correlated Tests

$$\Lambda(\mathbf{u}) = \frac{P(\mathbf{u}|H_1)}{P(\mathbf{u}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})} = \gamma$$

$$\Rightarrow \frac{P(u_1, u_2, \ldots, u_n|H_1)}{P(u_1, u_2, \ldots, u_n|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\Rightarrow \frac{\int_{u_1} \int_{u_2} \int_{\ldots} \int_{u_n} P_{x_1,\cdots,x_n}(X_1, \ldots, X_n|H_1)\, dx_1 \cdots dx_n}{\int_{u_1} \int_{u_2} \int_{\ldots} \int_{u_n} P_{x_1,\cdots,x_n}(X_1, \ldots, X_n|H_0)\, dx_1 \cdots dx_n} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

.

Combined Probability of Detection of Correlated Tests

$$
\begin{aligned}
P_D &= P(\Lambda(\mathbf{u}) \geq \gamma | H_1) \\
&= \sum_{\Lambda(\mathbf{u}) \geq \gamma} P(\mathbf{u}|H_1) \\
&= \sum_{(m_1,\cdots,m_z) \geq (k_1,\cdots,k_z)} \binom{n_1}{m_2} \cdots \binom{n_j}{m_j} \cdots \binom{n_z}{m_z} \\
&\quad \int_{u_1} \int_{u_2} \int_{\ldots} \int_{u_n} P_{X_1,\cdots,X_n}(X_1, \ldots, X_n | H_1)\, dx_1 \cdots dx_n \\
&= \sum_{(m_1,\cdots,m_z) \geq (k_1,\cdots,k_z)} \binom{n_1}{m_2} \cdots \binom{n_j}{m_j} \cdots \binom{n_z}{m_z} \int_{-\infty}^{\infty} \\
&\quad \prod_{j=1}^{z} \Phi\left(\frac{t - \mu_j - \sqrt{\rho} y}{\sqrt{\sigma^2 - \rho}} \right)^{(n_j - m_j)} \left[1 - \Phi\left(\frac{t - \mu_j - \sqrt{\rho} y}{\sqrt{\sigma^2 - \rho}} \right) \right]^{m_j} \phi(y)\, dy
\end{aligned}
$$

Combined False Alarm Rate of Correlated Tests

$$
\begin{aligned}
P_F &= P(\Lambda(\mathbf{u}) \geq \gamma | H_0) \\
&= \sum_{\Lambda(\mathbf{u}) \geq \gamma} P(\mathbf{u}|H_0) \\
&= \sum_{(m_1,\cdots,m_z) \geq (k_1,\cdots,k_z)} \binom{n_1}{m_2} \cdots \binom{n_j}{m_j} \cdots \binom{n_z}{m_z} \\
&\quad \int_{u_1} \int_{u_2} \int_{\ldots} \int_{u_n} P_{X_1,\cdots,X_n}(X_1,\ldots,X_n|H_0)\, dx_1 \,\cdots\, dx_n \\
&= \sum_{(m_1,\cdots,m_z) \geq (k_1,\cdots,k_z)} \binom{n_1}{m_2} \cdots \binom{n_j}{m_j} \cdots \binom{n_z}{m_z} \\
&\quad \int_{-\infty}^{\infty} \prod_{j=1}^{z} \Phi\left(\frac{t - \sqrt{\rho}y}{\sqrt{\sigma^2 - \rho}}\right)^{(n_j - m_j)} \left[1 - \Phi\left(\frac{t - \sqrt{\rho}y}{\sqrt{\sigma^2 - \rho}}\right)\right]^{m_j} \phi(y)\, dy
\end{aligned}
$$

Expected Cost

$$R = E(C) = \sum_{r=0}^{1}\sum_{s=0}^{1} C_{r,s} P(U_f = r, H_s)$$

$$= \sum_{r=0}^{1}\sum_{s=0}^{1} C_{r,s} P(U_f = r | H_s) P(H_s)$$

$$R^A \underset{A}{\overset{B}{\gtrless}} R^B$$

$$R^A - R^B = P(H_1)(C_{01} - C_{11})[P_D^B - P_D^A] + P(H_0)(C_{10} - C_{00})[P_F^A - P_F^B]$$

$$P_D = P(U_f = 1 | H_1) \text{ and } P_F = P(U_F = 1 | H_0)$$

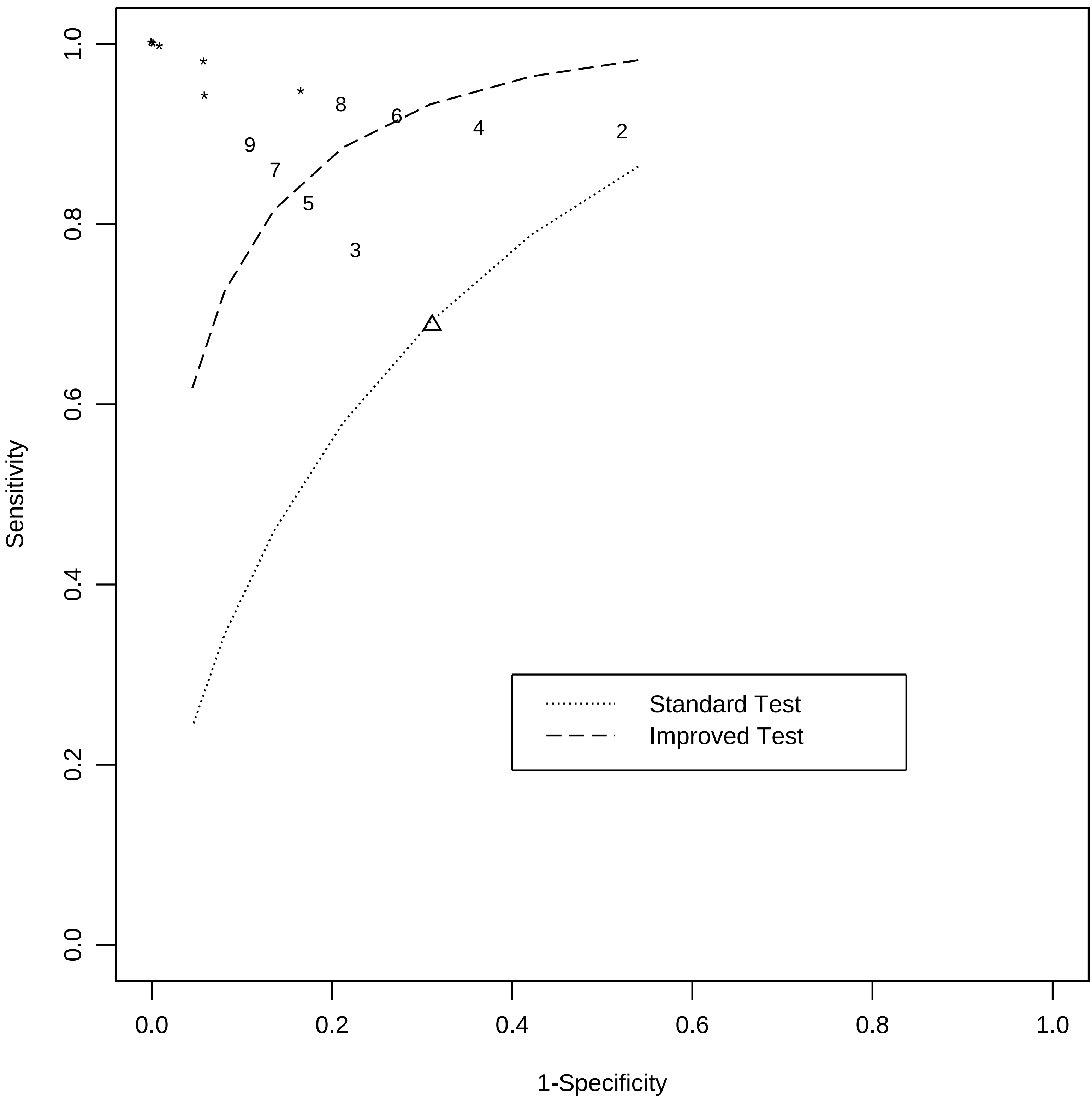
System Size and Correlation Consideration

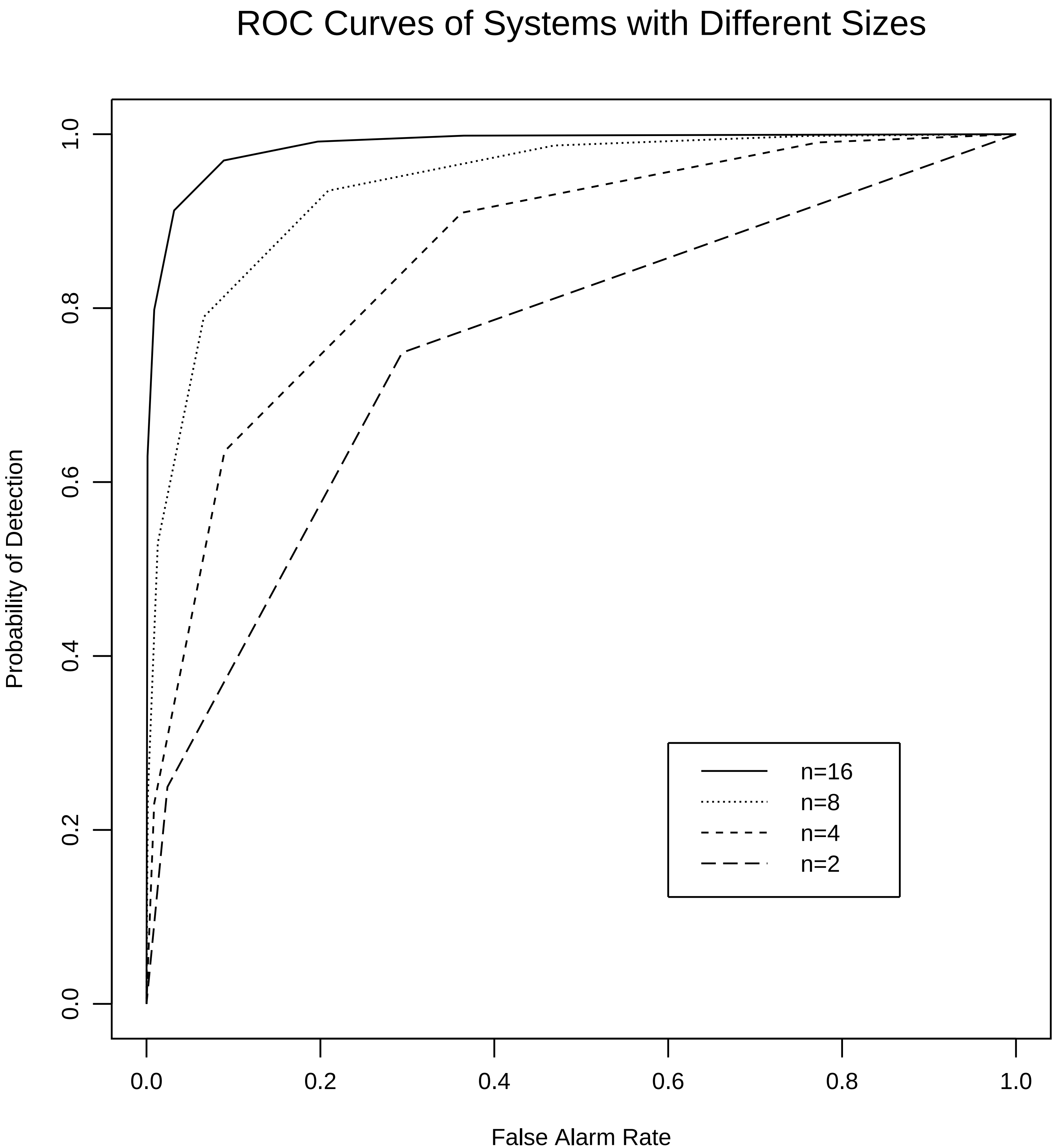
- Homogeneous individual tests (n=1-20), P_{D_i} and P_{F_i} are:
  - independent:(0.691,0.309), (0.460,0.135) and (0.864,0.540)
  - correlated: (0.691,0.309) and $\rho = 0, 0.2, 0.6$
- Heterogeneous individual tests
  - Type I test: $x_i|H_0 \sim N(0, \Sigma)$ and $x_i|H_1 \sim N(1, \Sigma)$
  - Type II test: $x_i|H_0 \sim N(0, \Sigma)$ and $x_i|H_1 \sim N(2, \Sigma)$

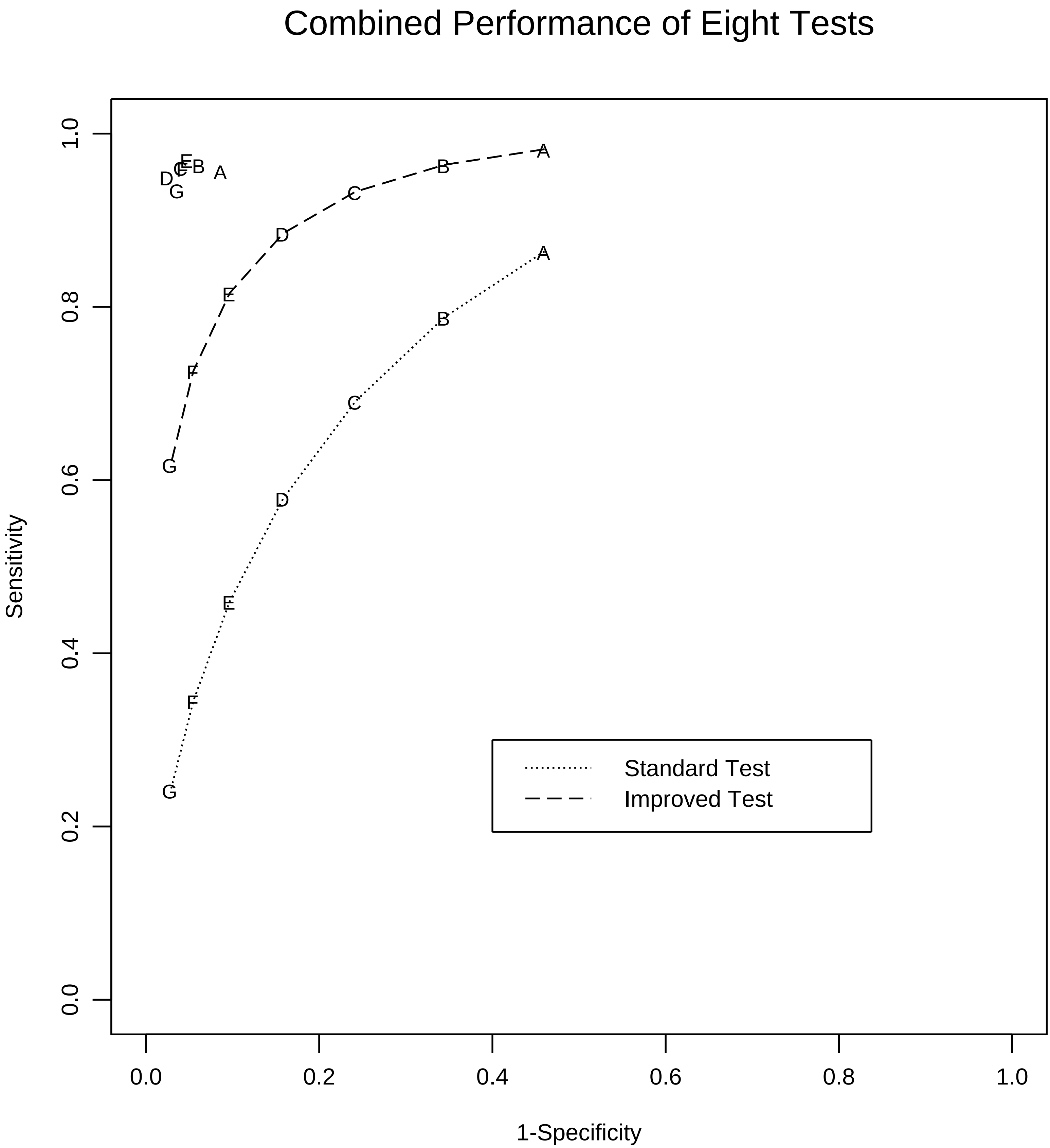
Combined Performance

Number of Tests	Number of Positive Tests Needed	Comb. Probability of Detection (P_D)	Comb. False Alarm Rates (P_F)
		$P_{Di} = 0.691$ *and* $P_{Fi} = 0.309$	
2	1	0.905	0.523
3	2	0.773	0.227
4	2	0.909	0.364
5	3	0.825	0.175
6	3	0.922	0.273
10	5	0.946	0.166
15	8	0.941	0.059
20	10	0.979	0.058

Combined Performance of Various Number of Regular Tests

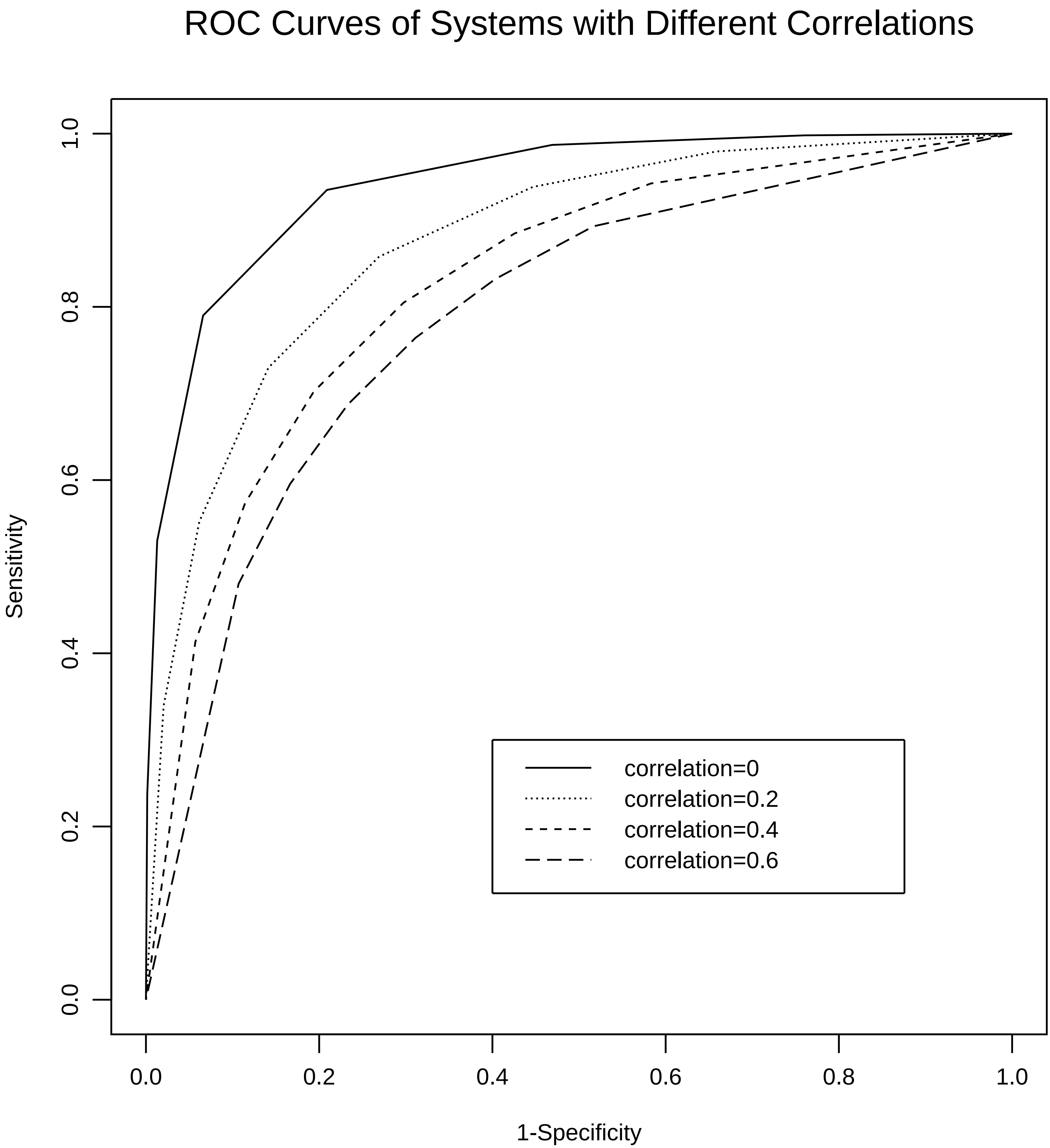



ROC Curves of Systems with Different Sizes
Probability of Detection
False Alarm Rate
0.0
0.2
0.4
0.6
0.8
1.0
n=16
n=8
n=4
n=2


Combined Performance of Eight Tests
Sensitivity
1-Specificity
0.0
0.2
0.4
0.6
0.8
1.0
Standard Test
Improved Test
A
B
C
D
E
F
G

Combined Performance

Test	Combined Prob. of Detection			Combined False Alarm Rates		
Number	$\rho = 0$	$\rho = 0.2$	$\rho = 0.6$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.6$
4	0.909	0.864	0.787	0.364	0.367	0.368
8	0.935	0.858	0.766	0.211	0.269	0.311
16	0.969	0.863	0.751	0.085	0.206	0.285
20	0.999	0.859	0.743	0.058	0.194	0.281


ROC Curves of Systems with Different Correlations
Sensitivity
1-Specificity
0.0
0.2
0.4
0.6
0.8
1.0
correlation=0
correlation=0.2
correlation=0.4
correlation=0.6

Example

- Objective: design an air pollution monitor network to monitor the level of nitrogen dioxide concentration exceeds some pre-specified level in Logan international Airport and surrounded community
- Pollution source: airport traffic
- Sampling strategy: judgmental sampling (end of runways, runway intersection, busy roadways in the airport and surrounded community)
- Spatial uniformity: annual coefficient of variation (standard deviation divided by the mean) is less than 10%
  - monitoring area, pollution source, sampling strategy

Example

- Type of sensors: Expensive (high resolution): $40,000 Continuous monitoring, low variance (5% of the mean) Inexpensive: $1,000 Passive, high variance (10% of the mean)
- Four different settings
  - Low cost monitor: (0.537, 0.417), (0.631, 0.417), (0.702, 0.417) and (0.755, 0.417)
  - expensive monitor: (0.566, 0.329), (0.672, 0.329), (0.747, 0.329) and (0.809, 0.329)
- Replacing 25% and 50% of the monitors in each system with high cost monitor

Performance of a System with Heterogenous Monitors

	Low	High	0%		25%		50%	
Number	P_{D1}	P_{D2}	P_D	P_F	P_D	P_F	P_D	P_F
16	0.631	0.672	0.798	0.177	0.855	0.153	0.867	0.115
32			0.910	0.129	0.933	0.076	0.950	0.046
64			0.961	0.013	0.981	0.020	0.991	0.009
16	0.702	0.747	0.829	0.077	0.895	0.073	0.907	0.056
32			0.934	0.033	0.968	0.024	0.980	0.015
64			0.987	0.007	0.996	0.003	0.999	0.001

Conclusions

- Increasing number of individual tests increases combined probability of detection (sensitivity) and decrease false alarm rate (1-specificity)
- Increasing correlations between individual tests decreases combined probability of detection and increase false alarm rate (specificity)
- Magnitude of declines due to correlations increases with the number of individual tests

Network Design Example

- Every receptor can only collect pollutant within certain radius and the pollutant concentrations measured at any receptors are emissions from nearby and distance sources
- Need to ensure that the distance between collocated samplers should be large enough to preclude the air sampled by any of the other monitors but small enough so that samplers obtaining the pollutants are emission from the target source
- In the situation where among detectors cannot be avoided, a larger number of individual tests is needed to compensate for loss

Limitations

- Temporal patterns
- Duration of concentrations
- Spatial uniformity failed
- Prob. of detection and false alarm rate of individual tests unknown
- Prior probability of phenomenon unknown