# Prediction for Max-Stable Processes via an Approximated Conditional Density


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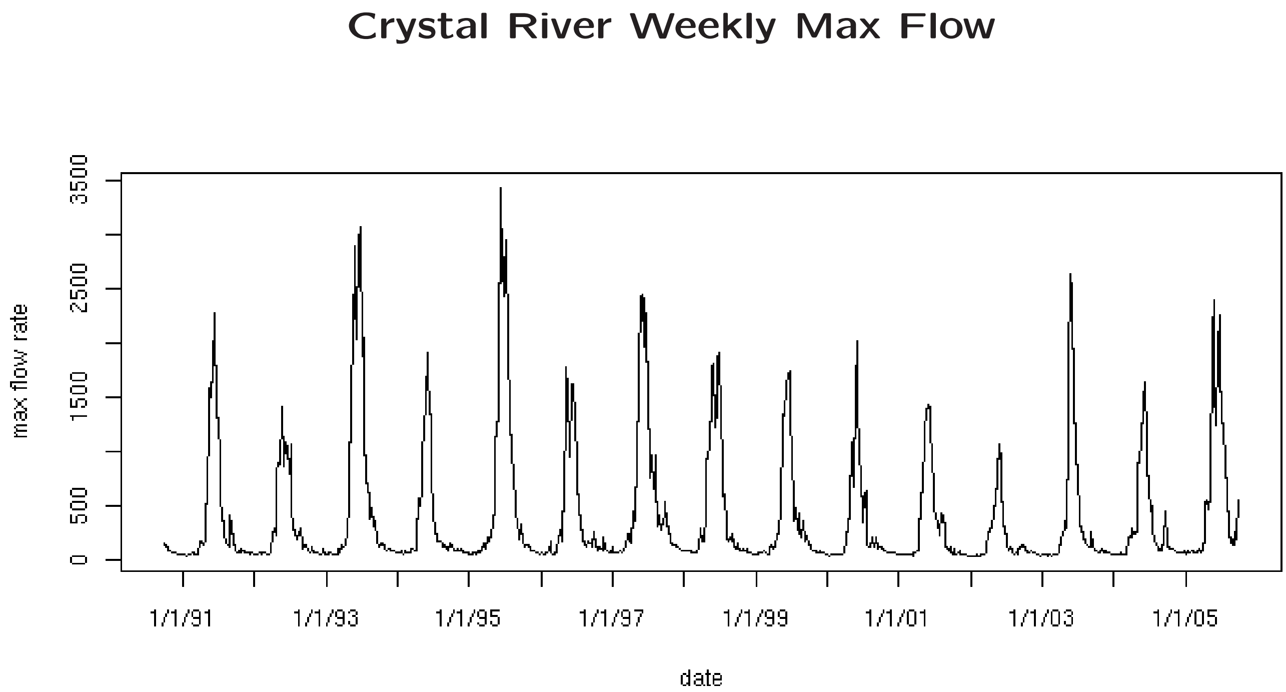
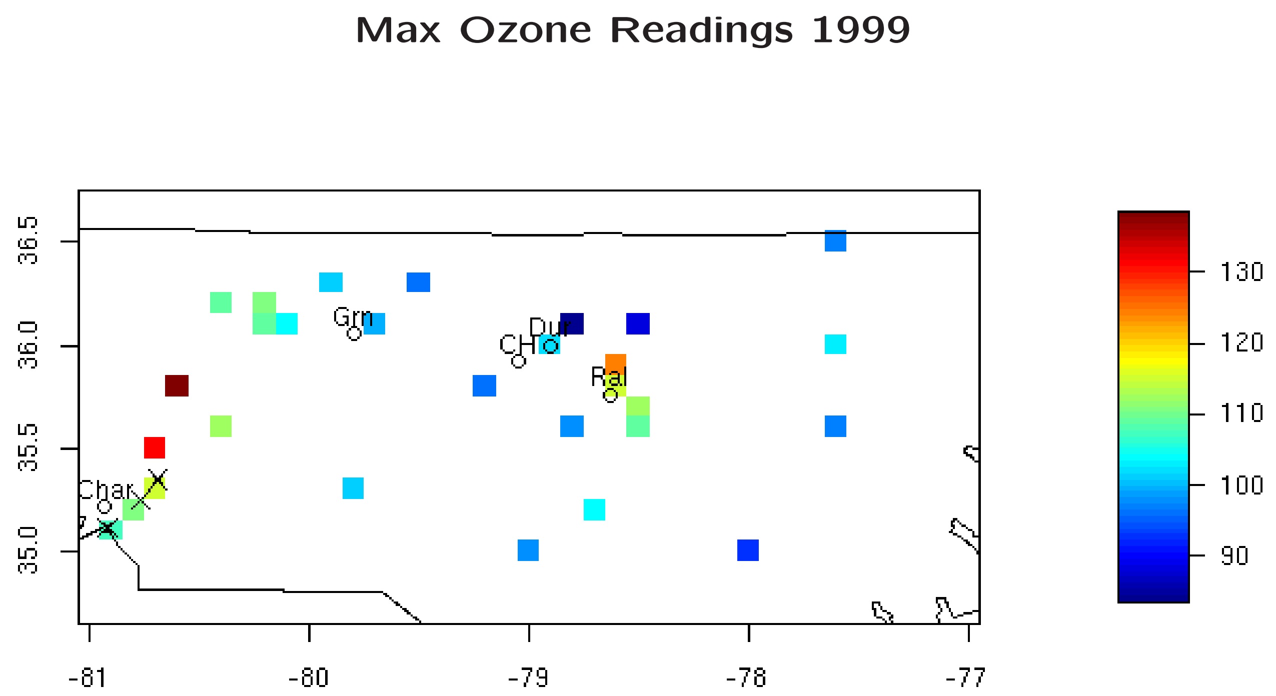
# Motivating Example 1: Time Series Data




Q: Can we use the previous weekly maximum flows to predict the next weekly maximum flow?

Q: What is a measure of risk (e.g., an estimated conditional 0.95 quantile) given that observed values are high ?

# Motivating Example 2: Spatial Data



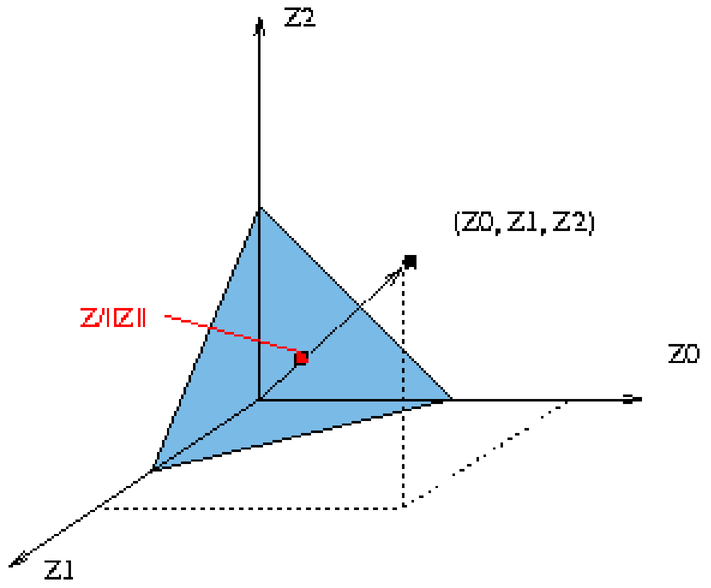

Q: Can we use the observed annual maxima to predict (interpolate) the annual max at an unobserved location?

Q: What is probability that it exceeds some standard?

# Our Approach for Project

1. Assume observations arise from a max-stable process from which we have a finite number of observations.
2. Find and fit a useful model for the observations plus the unmonitored location/time.
3. Approximate the *conditional density* of the unmonitored location given the “nearby” observations.

# Multivariate Max-Stable Distributions




*Basic idea:* Given a nice common marginal, MMSD's can be described in terms of a point process whose intensity measure is a product measure of "radial" and "angular" components.

# Multivariate Max-Stable Distributions

If $\boldsymbol{Z} = (Z(\boldsymbol{x}_1), \ldots, Z(\boldsymbol{x}_p))^T$ has a multivariate max-stable distribution with unit Fréchet margins ($\mathbb{P}(Z(\boldsymbol{x}_i) \leq z) = \exp(-z^{-1})$) then:

$$G(\boldsymbol{z}) = \mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}) = \exp[-V(\boldsymbol{z})], \text{ where}$$

$$V(\boldsymbol{z}) = \int_{S_p} \max_i \left( \frac{w_i}{z_i} \right) dH(\boldsymbol{w}),$$

$H$ is a positive measure on $S_p$, s.t.

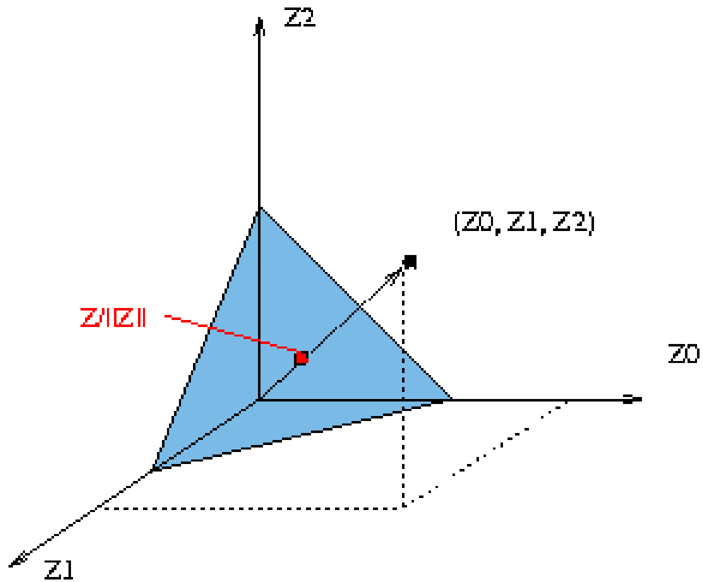
$$\int_{S_p} w_i dH(\boldsymbol{w}) = 1,$$

and $S_p = \{\boldsymbol{w} \in \mathbb{R}^p_+ | w_1 + \ldots + w_p = 1\}$.
Consequently,

$$\int_{S_p} dH(\boldsymbol{w}) = p$$

If $H$ is differentiable, then we have angular density $h(w)$, whose center of mass is $(1/p, 1/p, \ldots, 1/p)$.

# Multivariate Max-Stable Distributions



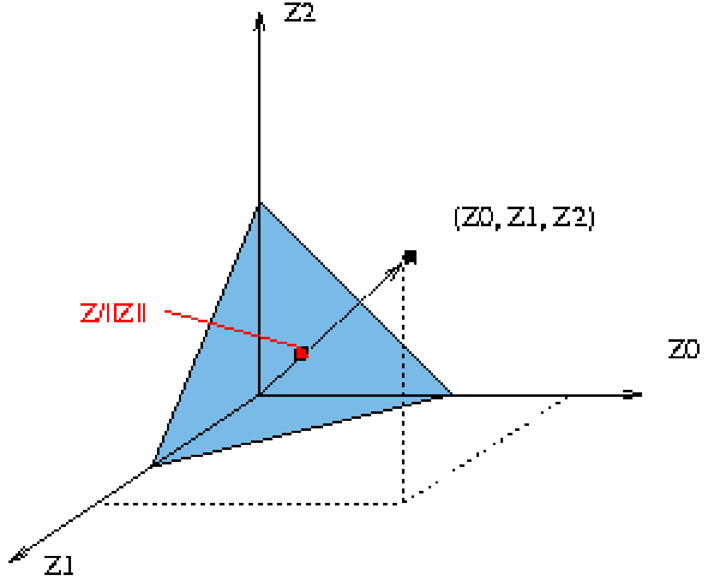

$\boldsymbol{z} \in \mathbb{R}^p; \boldsymbol{w} \in S_p$

$V(\boldsymbol{z})$: "exponent measure function" – relates to point process intensity, $G(\boldsymbol{z}) = \exp[-V(\boldsymbol{z})]$

$H(\boldsymbol{w})$: "angular measure" – lives on unit simplex, meets center of mass condition, $V(\boldsymbol{z}) = \int_{S_p} \max_i \left(\frac{w_i}{z_i}\right) dH(\boldsymbol{w})$

$h(\boldsymbol{w})$: "angular density" – exists if $H(\boldsymbol{w})$ is differentiable

# Multivariate Max-Stable Distributions




- no parametric form for entire family
- a few useful parametric sub-families suggested

# Models for Multivariate MSD's

| Exponent measure function $V(\boldsymbol{z})$ | Angular density $h(\boldsymbol{w})$ |
|---|---|
| • Logistic<br>• Asymmetric Logistic (Tawn, 88)<br>• Negative Logistic (Joe, 90) | • Dirichlet (Coles & Tawn, 91) |
| + Can obtain $G(\boldsymbol{z})$<br>– Overparametrized?<br>– Less flexible? | + More flexibility?<br>– Cannot directly get $G(\boldsymbol{z})$ |

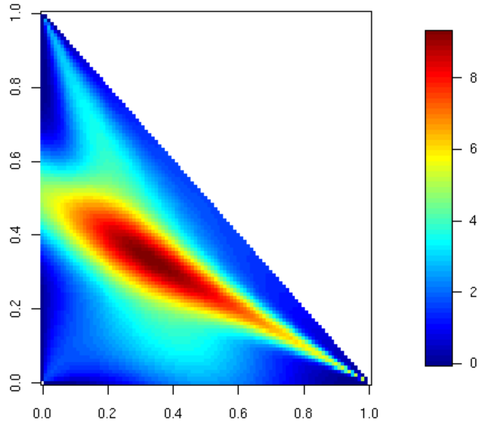
– Parameter interpretation?

# Pairwise Beta Model

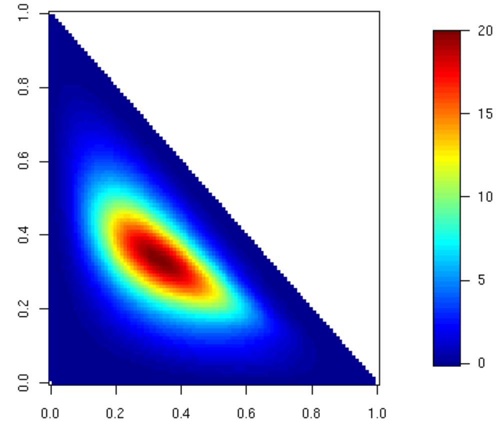
$$\begin{aligned}
h_p(\boldsymbol{w};\alpha,\boldsymbol{\beta}) &= K_p(\alpha)\sum_{i\neq j} h_{i,j}(w_i,w_j;\alpha,\beta_{i,j}), \text{ where} \\
h_{i,j}(w_i,w_j;\alpha,\beta_{i,j}) &= (w_i+w_j)^{(p-1)(\alpha-1)}(1-(w_i+w_j))^{\alpha-1}\times \\
&\quad \frac{\Gamma(2\beta_{i,j})}{(\Gamma(\beta_{i,j}))^2}\left(\frac{w_i}{w_i+w_j}\right)^{\beta_{i,j}-1}\left(\frac{w_j}{w_i+w_j}\right)^{\beta_{i,j}-1} \\
K_p(\alpha) &= \frac{2(p-3)!}{(p-1)\sqrt{p}}\frac{\Gamma(\alpha p)}{\Gamma(\alpha p-\alpha-p+3)\Gamma(\alpha+p-3)}
\end{aligned}$$

Advantages:

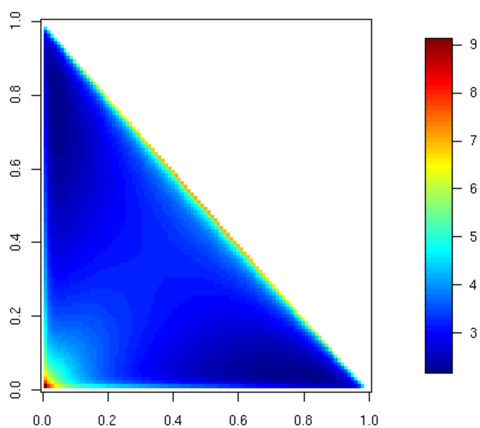
- no adjustment necessary to get center of mass condition
- parameters have some interpretation: $\alpha$ controls overall dependence, $\beta_{i,j}$'s control pairwise dependence
- largely specified by pairwise parameters



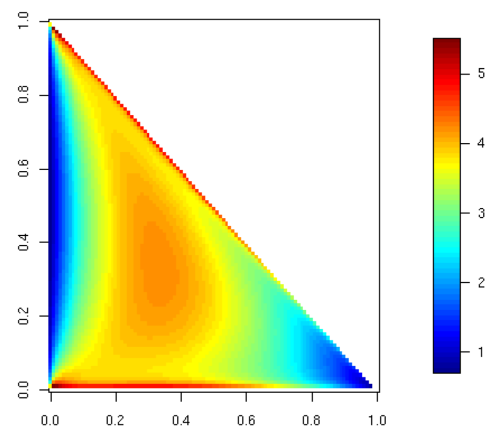

$\alpha = 1, \beta = (2, 4, 15)$




$\alpha = 4, \beta = (2, 4, 15)$




$\alpha = 1, \beta = (2, .5, .5)$



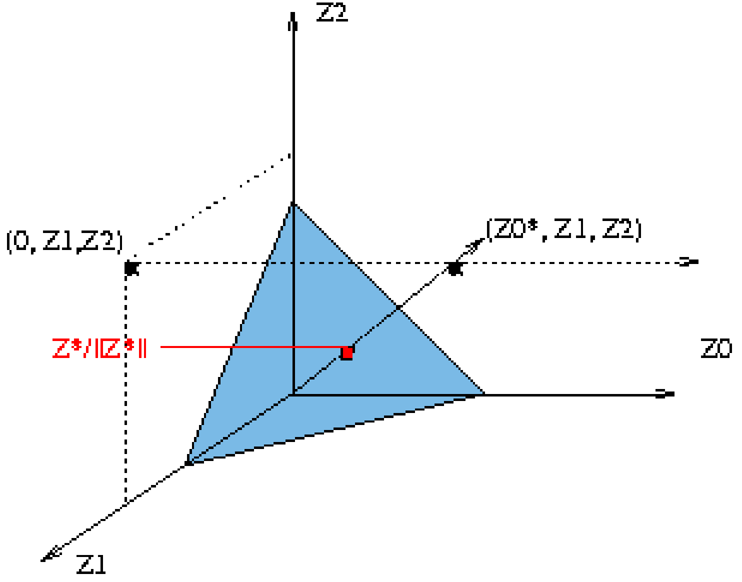

$\alpha = 1, \beta = (2, 2, .5)$

# Models for Multivariate MSD's

| Exponent measure function $V(\boldsymbol{z})$ | Angular density $h(\boldsymbol{w})$ |
|---|---|
| • Logistic<br>• Asymmetric Logistic (Tawn, 88)<br>• Negative Logistic (Joe, 90) | • Dirichlet (Coles & Tawn, 91)<br>• *Pairwise Beta*<br>* meets COM condition<br>* parameters interpretable<br>* pairwise specification |
| + Can obtain $G(\boldsymbol{z})$<br>– Overparametrized?<br>– Less flexible? | + More flexibility?<br>– Cannot directly get $G(\boldsymbol{z})$ |

– Parameter interpretation!

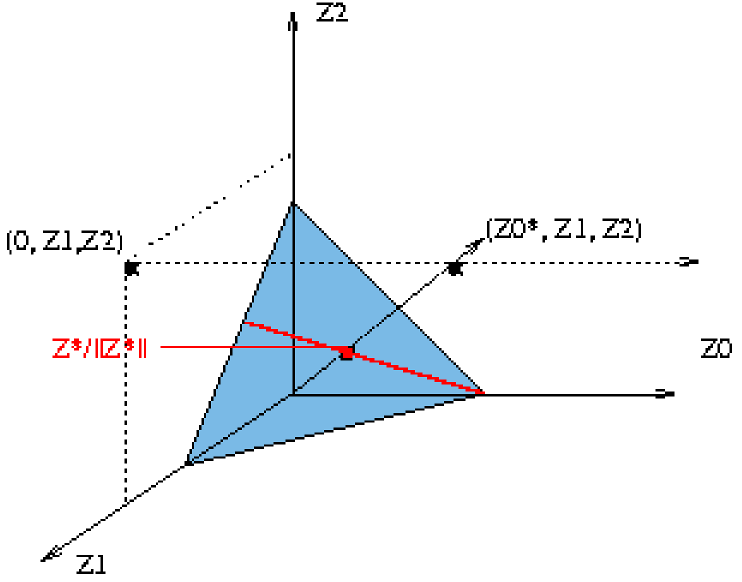
# Prediction: Approximating the conditional density?




If $V(\boldsymbol{z})$ is known and differentiable, then joint density can be obtained exactly. However, we are modeling $h(\boldsymbol{w})$.

Assume $Z_1, Z_2$ are observed and $Z_0$ is unobserved. Any predictor $Z_0^*$ will yield a point $\boldsymbol{Z}^* = (Z_0^*, Z_1, Z_2)$ which can be mapped back to $S_p$ as $\frac{\boldsymbol{Z}^*}{\|\boldsymbol{Z}^*\|_1}$.

# Approximating the conditional density?




Given the radius is large, by knowing the values of the angular density at $\dfrac{\boldsymbol{Z}^*}{\|\boldsymbol{Z}^*\|_1}$ and the value of the "radius" $\|\boldsymbol{Z}^*\|_1$, we can approximate the values of the joint density and in turn the *conditional density*.

# Approximating the conditional density?

If $V(\boldsymbol{z}) = \mu\{(0, z]^c\}$ is small (i.e. the radius is large), then

$$G(\boldsymbol{z}) = \exp(-V(\boldsymbol{z})) \approx 1 - V(\boldsymbol{z}).$$

Using Coles and Tawn (91) result to estimate the density at $\boldsymbol{z}$:

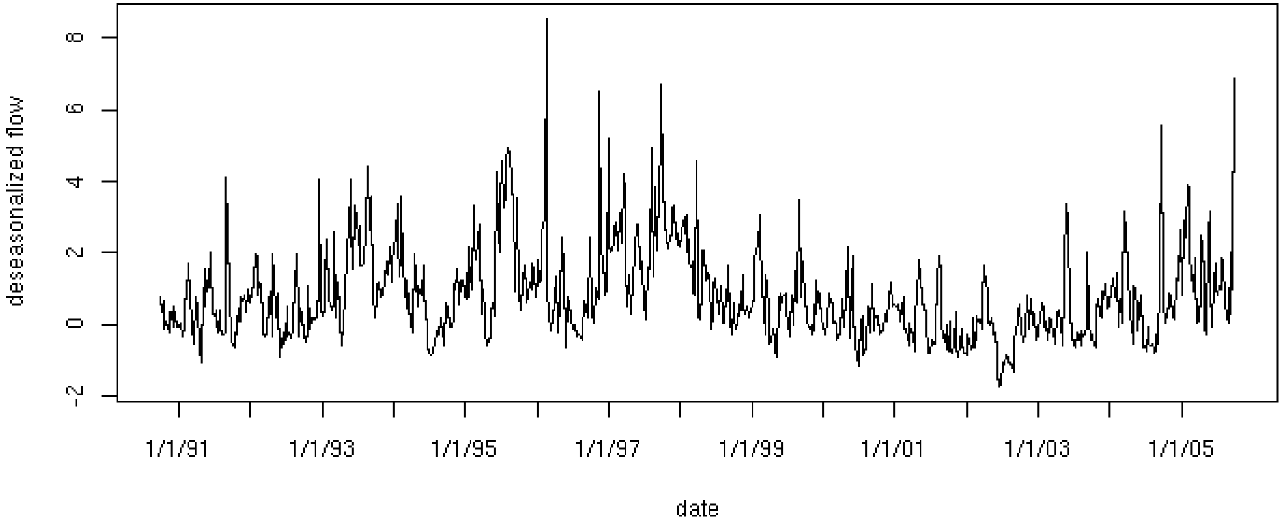
$$g(\boldsymbol{z}) \approx \frac{\partial}{\partial z_1, \ldots, \partial z_p}[1 - V(\boldsymbol{z})] = \frac{1}{||\boldsymbol{z}||^{-(p+1)}} h\left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)$$

So conditional density can be approximated by

$$g_{Z_p|Z_1,\ldots Z_{p-1}}(z_p|z_1, \ldots, z_{p-1}) \approx \frac{\frac{1}{||\boldsymbol{z}||^{-(p+1)}} h\left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)}{\int_0^\infty \frac{1}{||\boldsymbol{z}^*||^{-(p+1)}} h\left(\frac{\boldsymbol{z}^*}{||\boldsymbol{z}^*||}\right) d\zeta}$$

where $\boldsymbol{z}^* = (z_1, \ldots, z_{p-1}, \zeta)$.

# Time Series Example



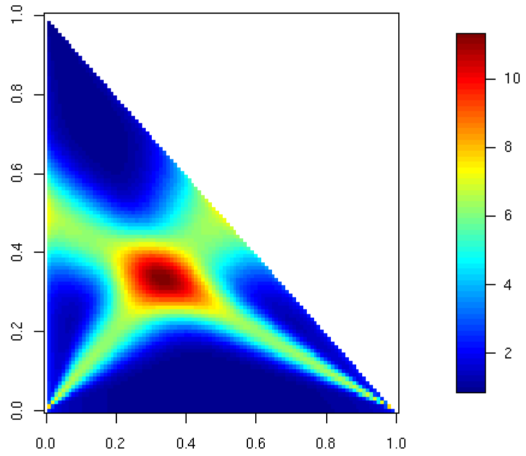

1. Data are deseasonalized.
2. ACF/PACF plots $\longrightarrow$ base prediction on previous two observations.
3. GEV is fit to marginal distribution, then transformed.

# Fitting the angular density model

Time series broken into non-overlapping triples. Dependence measured by the extremal coefficient (Schlather & Tawn 03).

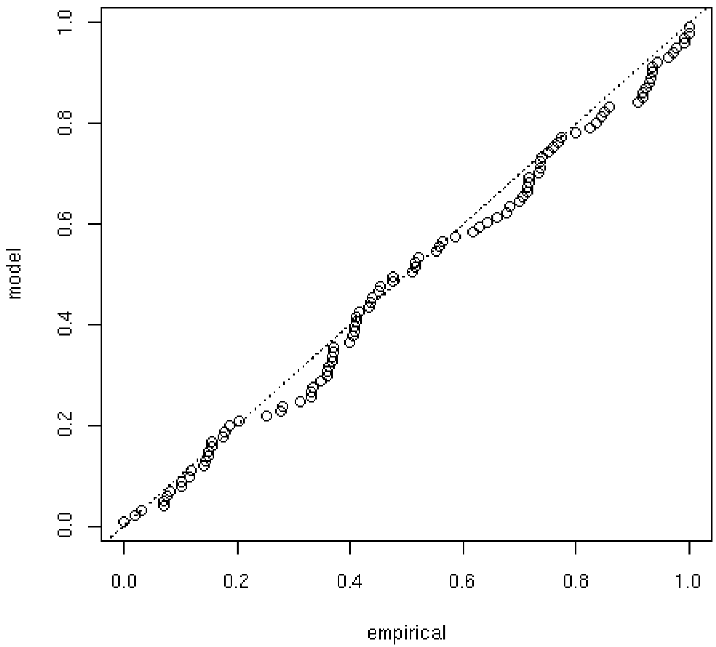
$$\phi_{1,2} = V(1, 1, \infty, \ldots, \infty); \ \phi_{i,j} \in [1, 2]$$

Ext. coefficients estimated at lags 1 and 2, $(\widehat{\phi}_{-1}, \widehat{\phi}_{-2}) = (1.36, 1.49)$, and pairwise beta model parameters found to match the extremal coefficient estimates. $(\widehat{\alpha}; \hat{\beta}) = (1, 16, 0.7, 16)$.



# Time series prediction

75 largest triples selected for prediction. Conditional density of 3rd component given 1st and 2nd components is approximated.



# Time series prediction

75 largest triples selected for prediction. Conditional density of 3rd component given 1st and 2nd components is approximated.

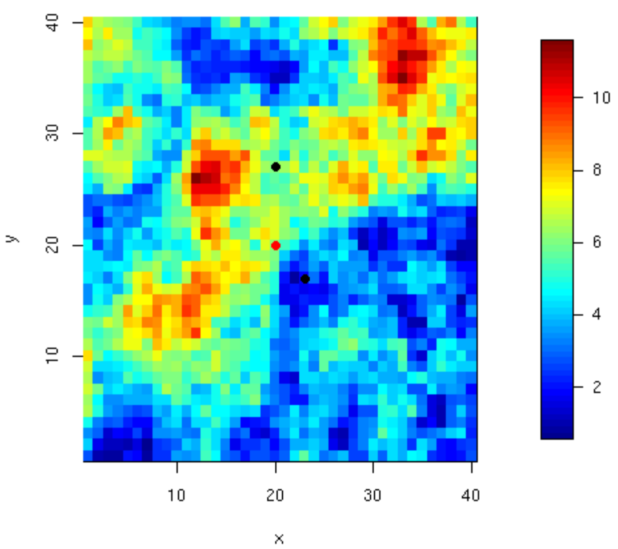
*Assessing Risk*

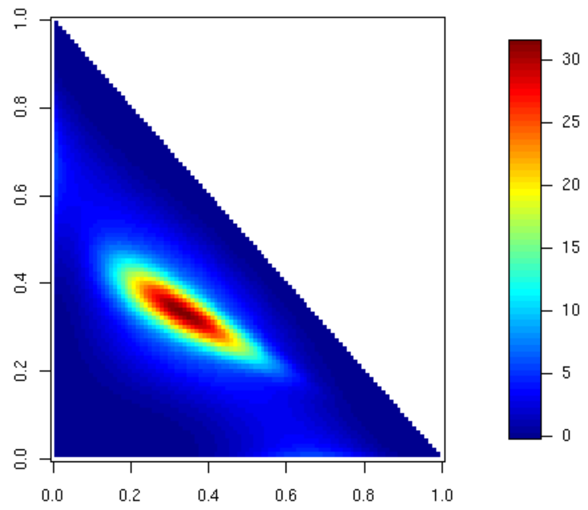
How well is the 95% quantile predicted?

- Our method: 6 observations (8%) exceed the estimated quantile
- AR(2): 9 observations (12%) exceed the estimated quantile

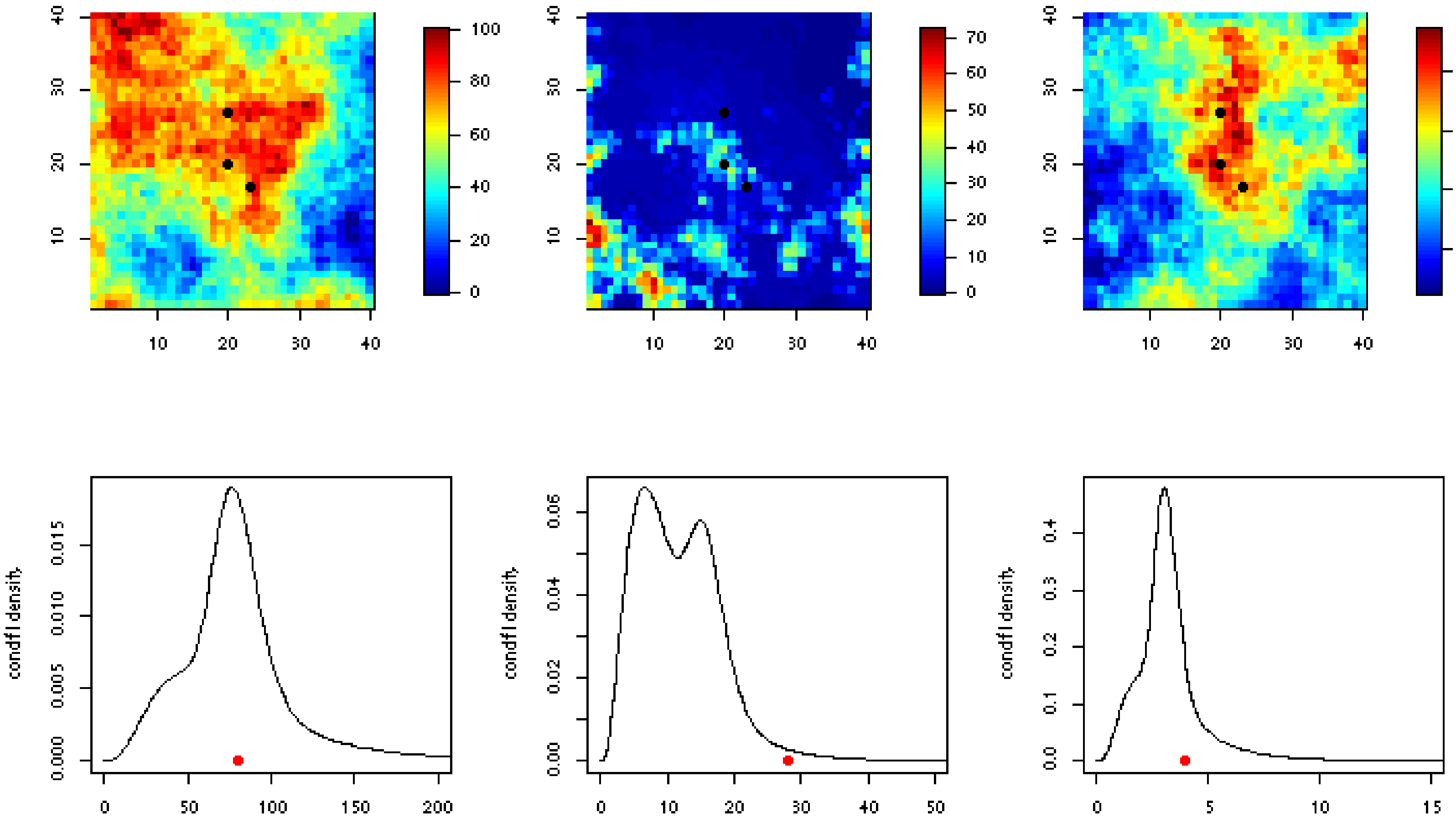
# Spatial interpolation

- MS fields w/ Fréchet marginals simulated (Schlather 02).
- Known bivariate dependence structure: $\phi = (1.34, 1.28, 1.22)$.
- Pairwise beta model fit as before: $(\alpha; \beta) = (4.3; 0.87, 4.4, 74)$.
- Conditional density approximated for largest 300 of 1000 simulated fields.



# Spatial interpolation examples

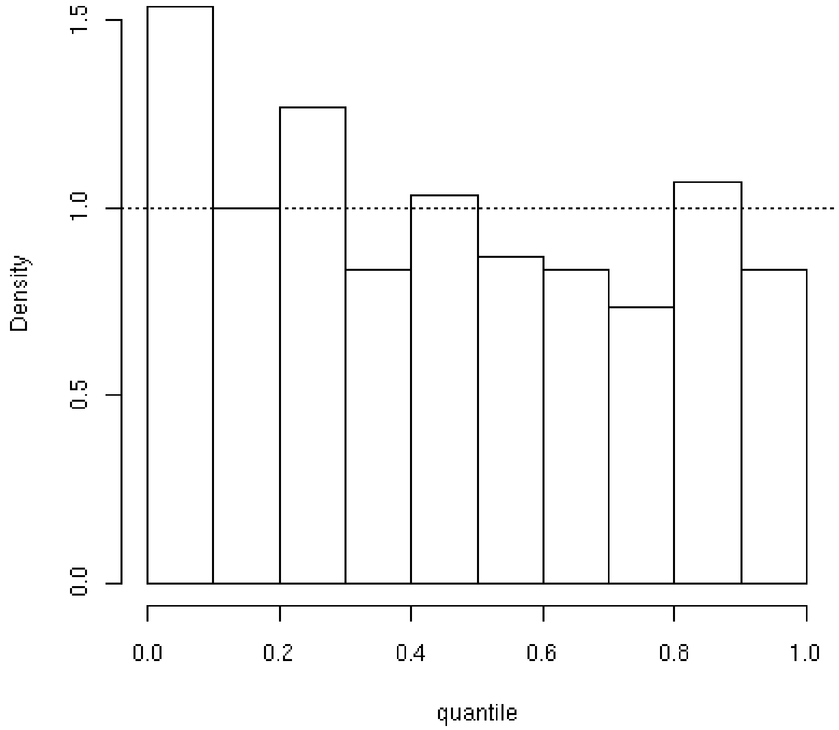



(80.69, 79.95, 80.45)

(4.50, 16.85, 28.14)

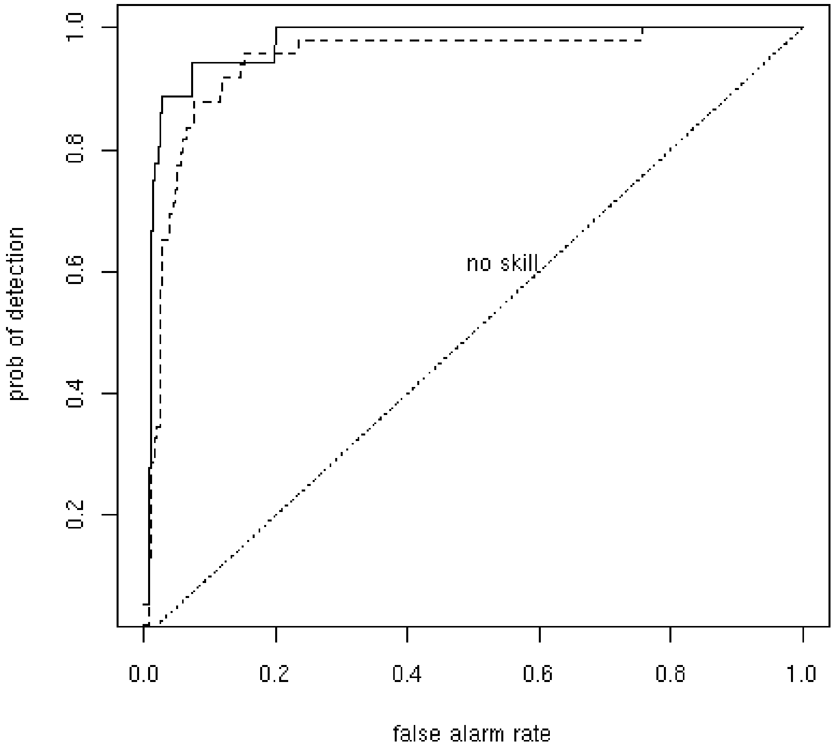
(3.47, 3.15, 3.93)

# Repeated Simulation Results

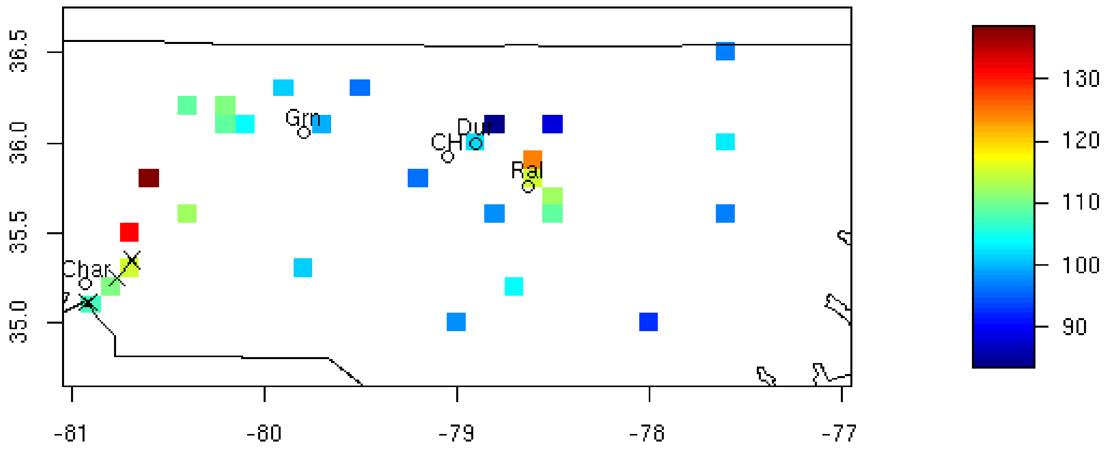


# Spatial interpolation results

How well does the method assess exceeding some standard?



# Ground level ozone

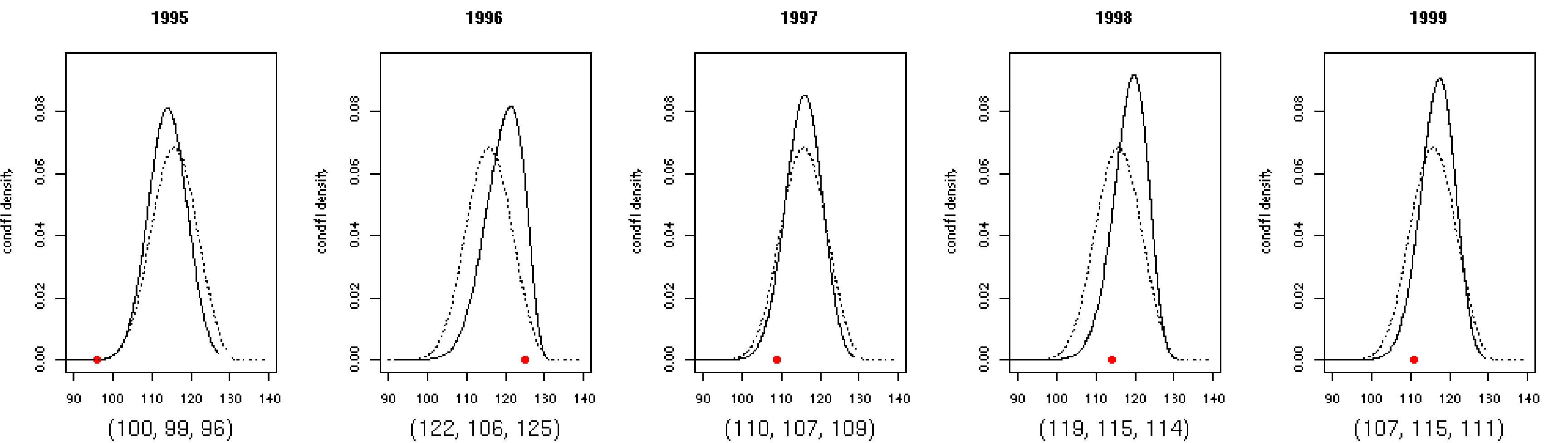



- Only 5 years of data.
- Marginal distributions from (Gilleland, et al 06).
- Dependence estimated as function of distance using madogram (Naveau et al 06).
- Weak dependence estimated $\phi = (1.95, 1.84, 1.67)$.

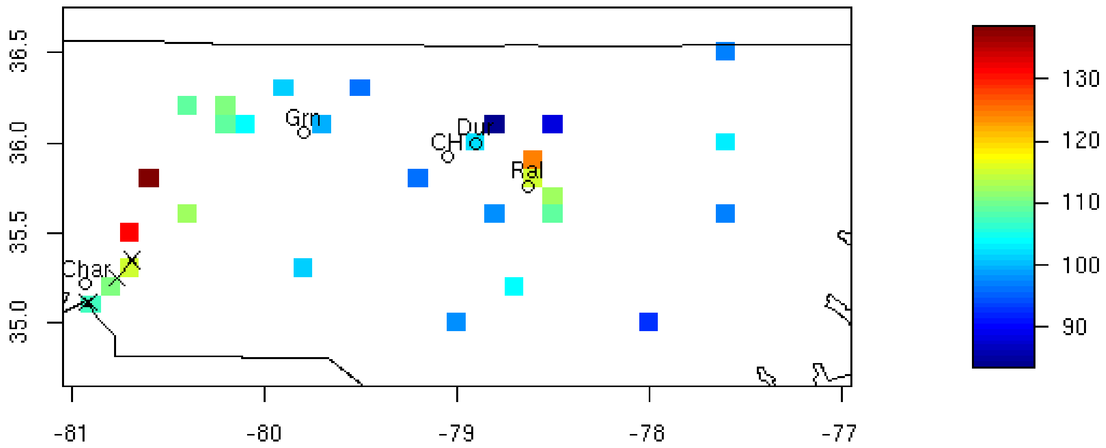
# Ground level ozone

Unable to restrict attention to the "large" years. An adjustment is made to earlier approximations.

$$g(\boldsymbol{z}) \approx \exp(-1/z_3) \frac{1}{||\boldsymbol{z}||^{(p+1)}} h\left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)$$



# So what?




Is this an artificial problem?

Our hope is to be able to extend this work to the more practical problem of prediction for exceedances or high observations.

# Summary

- Method for approximating the conditional density of an unobserved component of a max-stable vector given the other components.
- Method designed specifically for dealing with extremes; alternative to standard time series and spatial prediction methods which are better suited for central tendancies.
- Uses only the angular density, which (possibly) allows for more flexiblity in modeling.
- Applied to both time series and spatial contexts.
- Introduced pairwise beta model.
- Future work: extend from max-stable case to exceedances over a threshold case.

# References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