SMALL AREA ESTIMATION UNDER INFORMATIVE SAMPLING AND NONRESPONSE

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Introduction and Notation

$\{y_{ij}, x_{ij}; i = 1...M, j = 1...N_i\}$ - finite population measurements assumed to follow the **two level population model:**

$$y_{ij} \mid x_{ij}, u_i^U \sim f(y_{ij} \mid x_{ij}, u_i^U),\ i = 1...M,\ j = 1...N_i$$

$$u_i^U \sim f(u_i^U);\ E(u_i^U) = 0,\ V(u_i^U) = \sigma_{u^U}^2 .$$

y_{ij} - target study variable

$x_{ij} = (x_{ij}^1 ... x_{ij}^K)$ - covariates known for entire population.

Target: Estimate small area means $\bar{Y}_i = N_i^{-1} \sum_{j=1}^{N_i} y_{ij}$, based on the two-stage sample.

Two-stage Sampling Scheme:

Select m areas with inclusion probabilities $\pi_i = \Pr(i \in s)$,

Sample n_i units from selected cluster i with probabilities $\pi_{j|i} = \Pr(j \in s_i \mid i \in s)$.

$I_i, I_{ij} \rightarrow$ sample indicators,

$w_i = 1/\pi_i$, $w_{j|i} = 1/\pi_{j|i} \rightarrow$ sampling weights.

Unit non-response: $R_{ij} \rightarrow$ unit response indicators,

$R = \{(i, j) : I_i = 1, I_{ij} = 1, R_{ij} = 1\}$;

$R^c = \{(i, j) : I_i = 1, I_{ij} = 1, R_{ij} = 0\}$. **(No area non-response).**

Observed data

It is assumed that the response occurs independently between units.

The observed sample of respondents can be viewed therefore as the result of a two-phase sampling process where in the first phase the sample is selected from the population with **known** inclusion probabilities, and in the second phase the sample is 'self selected' with **unknown** response probabilities (Särndal and Swensson, 1987).

Model for observed data

Under our sampling scheme and response, the observed data follow the **two level respondents' model:**

$$y_{ij} \,|\, x_{ij}, u_i \sim f_{\boldsymbol{R}}(y_{ij} \,|\, x_{ij}, u_i) = f(y_{ij} \,|\, x_{ij}, u_i, (i,j) \in R),$$

$u_i \sim f(u_i \,|\, i \in s)$; $E(u_i \,|\, i \in s) = 0$, where $u_i = u_i^U - E(u_i^U \,|\, i \in s)$.

$f_{\boldsymbol{R}}(y_{ij} \,|\, x_{ij}, u_i) \neq f(y_{ij} \,|\, x_{ij}, u_i^U)$ **(population model)**

Since the model refers to the ***observed data***, it can be estimated and tested by classical SAE methods.

Let $p(y_{ij}, x_{ij}) = \Pr[(i, j) \in R \mid y_{ij}, x_{ij}, i \in s, j \in s_i]$.

If $p(y_{ij}, x_{ij})$ were known, the sample of respondents could be considered as a two-stage sample from the finite population with known selection probabilities π_i and $\tilde{\pi}_{j|i} = \pi_{j|i} p(y_{ij}, x_{ij})$.

Also, **if known**, the response probabilities could be used for imputation within the selected areas via the relationship between the **sample** and **sample-complement distributions** (Sverchkov & Pfeffermann, 2004)**;**

$$f(y_{ij} \mid x_{ij}, u_i, (i,j) \in R^c) =$$

$$\frac{[p^{-1}(y_{ij}, x_{ij}) - 1] f(y_{ij} \mid x_{ij}, u_i, (i,j) \in R)}{E\{[p^{-1}(y_{ij}, x_{ij}) - 1] \mid x_{ij}, u_i, (i,j) \in R\}}. \qquad \textbf{(1)}$$

(1) refers to the model for the ***observed*** data and therefore can be estimated by classical SAE methods.

Estimation of response probabilities

Assume a parametric model for the **response probabilities** $p(y_{ij}, x_{ij}; \gamma) = \Pr[(i, j) \in R \mid y_{ij}, x_{ij}, i \in s, j \in s_i; \gamma]$ and suppose that p is differentiable with respect to the (**vector**) parameter γ.

If the missing data were observed, γ could be estimated by solving the equations:

$$0 = \sum_{(i,j)\in R} \frac{\partial \log p(y_{ij}, x_{ij}; \gamma)}{\partial \gamma} + \sum_{(i,j)\in R^c} \frac{\partial \log[1 - p(y_{ij}, x_{ij}; \gamma)]}{\partial \gamma}. \qquad \textbf{(2)}$$

Denote the observed data by

$\boldsymbol{O} = \{y_{ij}, \pi_{j|i}, \pi_i, n_i, (i, j) \in R;\ x_{kl}, k = 1...M,\ l = 1...N_i\}$.

Missing Information Principle: since the outcome values are missing for $(i,j) \in R^c$, we propose to solve instead,

$$0 = E\{[\sum_{(i,j)\in R} \frac{\partial \log p(y_{ij}, x_{ij}; \gamma)}{\partial \gamma} + \sum_{(i,j)\in R^c} \frac{\partial \log[1 - p(y_{ij}, x_{ij}; \gamma)]}{\partial \gamma}] \mid O\} =$$

$$\sum_{(i,j)\in R} \frac{\partial \log p(y_{ij}, x_{ij}; \gamma)}{\partial \gamma} +$$

$$\sum_{(i,j)\in R^c} E\{\frac{\partial \log[1 - p(y_{ij}, x_{ij}; \gamma)]}{\partial \gamma} \mid O, (i,j) \in R^c\} \overset{\text{by (1)}}{=}$$

$$\sum_{(i,j)\in R} \frac{\partial \log p(y_{ij}, x_{ij};\gamma)}{\partial \gamma} +$$

$$\sum_{(i,j)\in R^c} E\left(\frac{E\{[p^{-1}(y_{ij}, x_{ij};\gamma)-1]\frac{\partial \log[1-p(y_{ij}, x_{ij};\gamma)]}{\partial \gamma} \mid x_{ij}, u_i, (i,j)\in R\}}{E\{[p^{-1}(y_{ij}, x_{ij};\gamma)-1] \mid x_{ij}, u_i, (i,j)\in R\}} \middle| O \right) = 0 \quad \textbf{(3)}$$

(we assume $f(y_{ij} \mid O, u_i, (i,j)\in R) = f(y_{ij} \mid x_{ij}, u_i, (i,j)\in R)$)

The expectations in **(3)** refer to the model for the ***observed*** data and therefore can be estimated by classical SAE methods.

The parameter γ can be estimated by solving (3).

Note: if $p(y_{ij}, x_{ij}; \gamma)$ is a function of x_{ij} and γ only, (**missing data are MAR**), **(3)** reduces to the common log-likelihood equations,

$$0 = \sum_{(i,j)\in R} \frac{\partial \log p(x_{ij};\gamma)}{\partial \gamma} + \sum_{(i,j)\in R^c} \frac{\partial \log[1 - p(x_{ij};\gamma)]}{\partial \gamma}. \quad \textbf{(4)}$$

Prediction of small area means (P-S, *JASA* 2007)

$$MSE(\hat{\bar{Y}}_i) = E[(\hat{\bar{Y}}_i - \bar{Y}_i)^2 \mid O, I_i) = [\hat{\bar{Y}}_i - E(\bar{Y}_i \mid O, I_i)]^2 + V(\bar{Y}_i \mid O, I_i)$$

$\hat{\bar{Y}}_i = E(\bar{Y}_i \mid O, I_i)$ - Optimal small area predictor for area i.

Optimal small-area predictors for selected areas:

$$\hat{\bar{Y}}_i = E(\bar{Y}_i \mid O, I_i = 1) = N_i^{-1}[\sum_{j:(i,j)\in R} y_{ij} + \sum_{k=1,k\notin R}^{N_i} E(y_{ik} \mid O, I_i = 1)] \cong$$

$$N_i^{-1}(\sum_{j,(i,j)\in R} y_{ij} +$$

$$\sum_{k=1,k\notin R}^{N_i} E\{\frac{E[(\tilde{\pi}_{k|i}^{-1} - 1) y_{ik} \mid x_{ik}, u_i, (i,k)\in R]}{E[(\tilde{\pi}_{k|i}^{-1} - 1) \mid x_{ik}, u_i, (i,k)\in R]} \mid O\}) \cong$$

$$N_i^{-1}\Big(\sum_{j,(i,j)\in R} y_{ij} +$$

$$\sum_{k=1,k\notin R}^{N_i} E\{\frac{E\{[w(y_{ik},x_{ik})-1]y_{ik} \mid x_{ik},u_i,(i,k)\in R\}}{E\{[w(y_{ik},x_{ik})-1] \mid x_{ik},u_i,(i,k)\in R\}} \mid O\}\Big)\textbf{;} \quad \textbf{(5)}$$

$\hat{\tilde{\pi}}_{k|i} = \pi_{k|i} p(y_{ik},x_{ik};\hat{\gamma})$ and $w(y_{ik},x_{ik}) = E[\hat{\tilde{\pi}}_{k|i}^{-1} \mid y_{ik},x_{ik},(i,k)\in R]$.

(Refers to **observed data** and can be estimated by regression or non-parametrically).

Expectations in **(5)** are over the model for the ***observed*** data that was **estimated before**.

Optimal small-area predictors for unselected areas:

$$\hat{\bar{Y}}_i = E(\bar{Y}_i \mid O, I_i = 0) = N_i^{-1}[\sum_{k=1}^{N_i} E(y_{ik} \mid O, I_i = 0)]$$

$$\cong N_i^{-1} \sum_{k=1}^{N_i} \frac{\sum_{l \in s} [(\pi_l^{-1} - 1) K_l(x_{ik})]}{\sum_{l \in s} (\pi_l^{-1} - 1)} \qquad \textbf{(6)}$$

$$K_l(x) = E(y_{lk} \mid x_{lk} = x, (l,k) \in U) =$$

$$E\{\frac{E[w(y_{lk}, x_{lk}) y_{lk} \mid x_{lk} = x, u_l, (l,k) \in R]}{E[w(y_{lk}, x_{lk}) \mid x_{lk} = x, u_l, (l,k) \in R]} \mid O\}$$

(6) depends on $w(y_{lk}, x_{lk})$ and the model for the ***observed*** data.

Example: Logistic Mixed Model with Logistic Response

Let $y_{ij} \sim Bernoulli$.

Working model for **observed data** (**can be identified and tested**):

$$\Pr(y_{ij} = 1 \mid x_{ij}, u_i, (i,j) \in R) = p_y(x_{ij}, u_i) = \frac{\exp(\beta_0 + \beta_1 x_{ij} + u_i)}{1 + \exp(\beta_0 + \beta_1 x_{ij} + u_i)},$$

$$u_i \sim N(0, \sigma_u^2).$$

Working response model (**has to be assumed**):

$$p(y_{ij}, x_{ij}, \gamma) = \frac{\exp(\gamma_0 + \gamma_1 x_{ij} + \gamma_2 y_{ij})}{1 + \exp(\gamma_0 + \gamma_1 x_{ij} + \gamma_2 y_{ij})}.$$

The first expectation in **(3)** can be written as

$$E\{[p^{-1}(y_{ij},x_{ij};\gamma)-1]\frac{\partial \log[1-p(y_{ij},x_{ij};\gamma)]}{\partial\gamma}\,|\,x_{ij},u_i,(i,j)\in R\}=$$

$$p_y(x_{ij},u_i)[p^{-1}(1,x_{ij};\gamma)-1]\frac{\partial \log[1-p(1,x_{ij};\gamma)]}{\partial\gamma}+$$

$$[1-p_y(x_{ij},u_i)][p^{-1}(0,x_{ij};\gamma)-1]\frac{\partial \log[1-p(0,x_{ij};\gamma)]}{\partial\gamma}.$$

- Similarly for the second expectation in **(3)**.

$p_y(x_{ij})$ and $\hat{u}_i$ easily estimated by **SAS PROC NLMIX** and **(3)** is solved for γ by **SAS PROC NLIN**.

$$E[(w(y_{ij},x_{ij})-1)y_{ij}\,|\,x_{ij},u_i,(i,j)\in R]=p_y(x_{ij},u_i)(w(1,x_{ij})-1).$$

- Similarly for other expectations in **(5)** and **(6)**.

Simulation Study

Step 1: Generate finite population from **Population model:**

$y_{ij} \sim Bernoulli$,

$$\Pr(y_{ij}=1 \mid x_{ij}, u_i^U, (i,j)\in R) = p_y(x_{ij}, u_i^U) = \frac{\exp(-1 + x_{ij} + u_i^U)}{1+\exp(-1+x_{ij}+u_i^U)},$$

$u_i^U \sim N(0,1)$.

$M = 300,\ N_i = \text{int}[1000\exp\{\min[2.5, \max(\text{-}2{,}5, u_i^U)]/5\}]$,

$x_{ij} \sim Uniform(0,2)$.

Group areas into **3 sets**,

G1={i=1,..,100}, G2={i=101,..,200}, G3={i=201,..,300}.

Step 2: Sampling scheme:

Select m=150 areas by systematic PPS proportional to area size N_i (**informative sampling**).

Select **20** units from each selected area in G1,

40 units from each selected area in G2,

60 units from each selected area in G3,

by PPS sampling proportional to $z_{ij} = .5 + x_{ij} + 3y_{ij}$ (**informative sampling**).

Step 3: Response:

Each selected unit responds with probability

$$p(y_{ij}, x_{ij}, \gamma) = \frac{\exp(-.5x_{ij} + y_{ij})}{1 + \exp(-.5x_{ij} + y_{ij})}.$$

Step 4: Estimate $\hat{p}_y(x_{ij}, \hat{u}_i) = \hat{\Pr}(y_{ij} = 1 | x_{ij}, \hat{u}_i, (i, j) \in R)$ assuming **Logistic Mixed Model** for the respondents, applying PROC NLMIX with default options (Empirical Bayes).

Step 5: Assume working response model,

$$p(y_{ij}, x_{ij}, \gamma) = \frac{\exp(\gamma_0 + \gamma_1 x_{ij} + \gamma_2 y_{ij})}{1 + \exp(\gamma_0 + \gamma_1 x_{ij} + \gamma_2 y_{ij})}$$

. Substitute $\hat{p}_y(x_{ij}, \hat{u}_i)$ into **(3)** and estimate γ by use of **PROC NLIN**.

Estimate $w(y_{ij}, x_{ij}) = E[\tilde{\pi}_{j|i}^{-1} \mid y_{ij}, x_{ij}, (i,j) \in R]$ as follows:

$$E[\tilde{\pi}_{j|i}^{-1} \mid y_{ij}, x_{ij}, (i,j) \in R] = p(y_{ij}, x_{ij}) E[\pi_{j|i}^{-1} \mid y_{ij}, x_{ij}, (i,j) \in R];$$

$$\pi_{j|i} = n_i z_{ij} / \sum_{j=1}^{N_i} z_{ij} = \frac{n_i}{N_i} z_{ij} \underbrace{\left(N_i / \sum_{j=1}^{N_i} z_{ij} \right)}_{\approx \text{Constant}} \Rightarrow z_{ij}^{*} = \pi_{j|i} \frac{N_i}{n_i} \prec z_{ij}.$$

Fit the model $z_{ij}^{*} = g_{\alpha}(y_{ij}, x_{ij})$ (linear model in our study), estimate the parameters of this model and then estimate,

$$\hat{w}(y_{ij}, x_{ij}) = \hat{E}[\tilde{\pi}_{j|i}^{-1} \mid y_{ij}, x_{ij}, (i, j) \in R] \cong$$

$$\left[\frac{n_i}{N_i} g_{\hat{\alpha}}(y_{ij}, x_{ij}) \right]^{-1} p(y_{ij}, x_{ij}; \hat{\gamma}).$$

Calculate ratio of expectations in **(5),**

$$\hat{p}_y^{R^c}(x_{ik}, \hat{u}_i) = \hat{E}\{\frac{\hat{E}[(\hat{w}(y_{ik}, x_{ik}) - 1) y_{ik} \mid x_{ik}, u_i, (i,k) \in R]}{\hat{E}[(\hat{w}(y_{ik}, x_{ik}) - 1) \mid x_{ik}, u_i, (i,k) \in R]} \mid O\}.$$

Estimators considered (selected areas):

1. $\hat{\bar{Y}}_i^{ign} = N_i^{-1}\{ \sum_{j,(i,j)\in R} y_{ij} + \sum_{k=1,k\notin R}^{N_i} \hat{p}_y(x_{ij}) \}$

2. $\hat{\bar{Y}}_i^{H,MCAR} = \sum_{j,(i,j)\in R} \pi_{j|i}^{-1} y_{ij} \,/ \sum_{j,(i,j)\in R} \pi_{j|i}^{-1}$

3. $\hat{\bar{Y}}_i^{H,MAR} = \sum_{j,(i,j)\in R} \hat{w}(x_{ij}) y_{ij} \,/ \sum_{j,(i,j)\in R} \hat{w}(x_{ij}),\quad \hat{w}(x_{ij}) = [\pi_{j|i} p(x_{ij}, \hat{\lambda})]^{-1},$

4. $\hat{\bar{Y}}_i^{H,new} = \sum_{j,(i,j)\in R} \hat{w}(y_{ij}, x_{ij}) y_{ij} \,/ \sum_{j,(i,j)\in R} \hat{w}(y_{ij}, x_{ij}),$

5. $\hat{\bar{Y}}_i^{new} = N_i^{-1}\{ \sum_{j,(i,j)\in R} y_{ij} + \sum_{k=1,k\notin R}^{N_i} \hat{p}_y^{R^c}(x_{ij}, \hat{u}_i) \}.$

Repeat Steps 1-5 independently 1000 times.

Statistics considered:

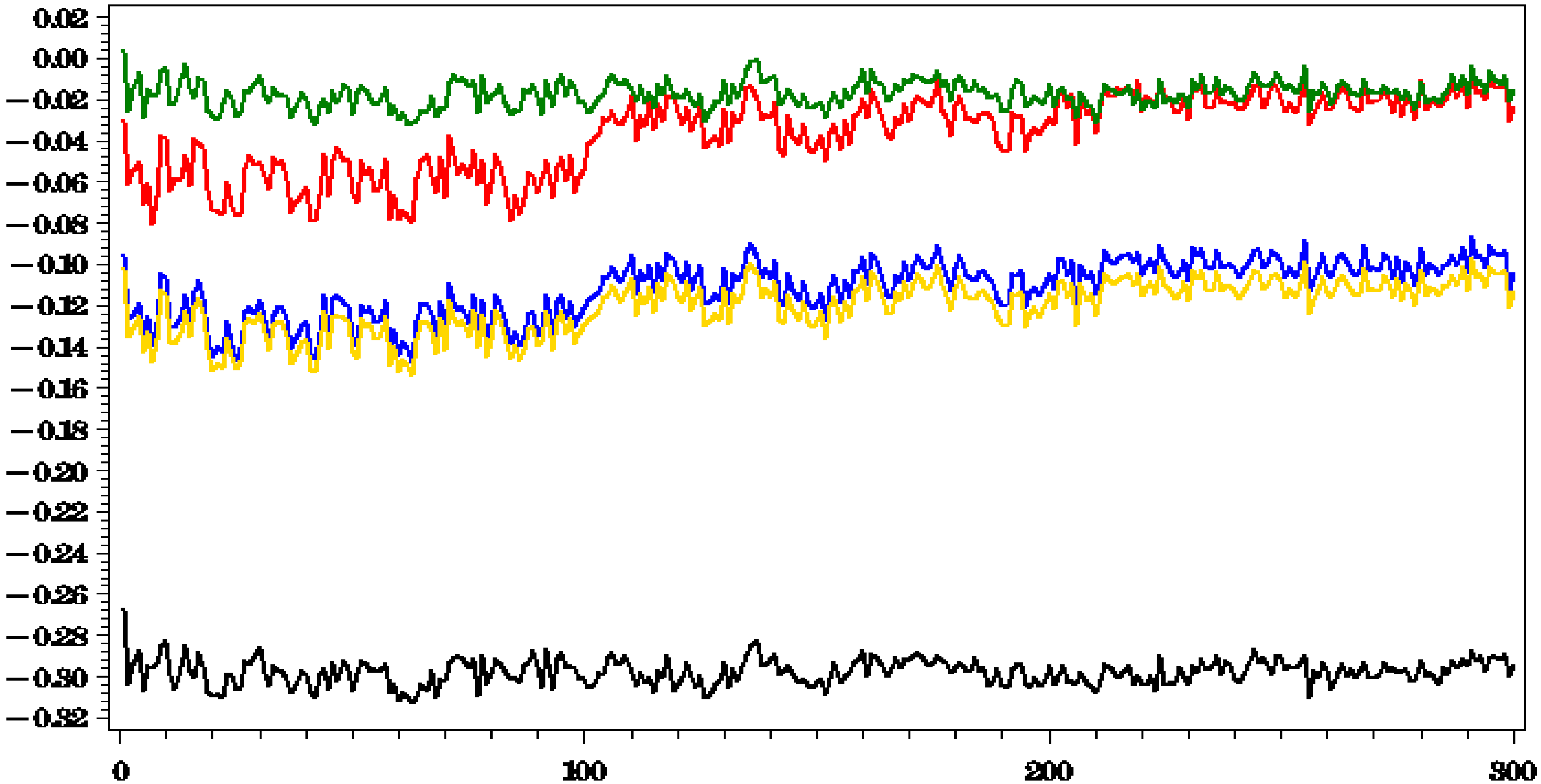
$$Bias_i = \frac{\sum_{r=1}^{1000} D_{ir}(\hat{\bar{Y}}_{ir} - \bar{Y}_{ir})}{\sum_{r=1}^{1000} D_{ir}}$$

$$RMSE_i = \sqrt{\frac{\sum_{r=1}^{1000} D_{ir}(\hat{\bar{Y}}_{ir} - \bar{Y}_{ir})^2}{\sum_{r=1}^{1000} D_{ir}}}$$

$D_{ir} = 1$ if area i **selected** on ***r*-th** simulation.

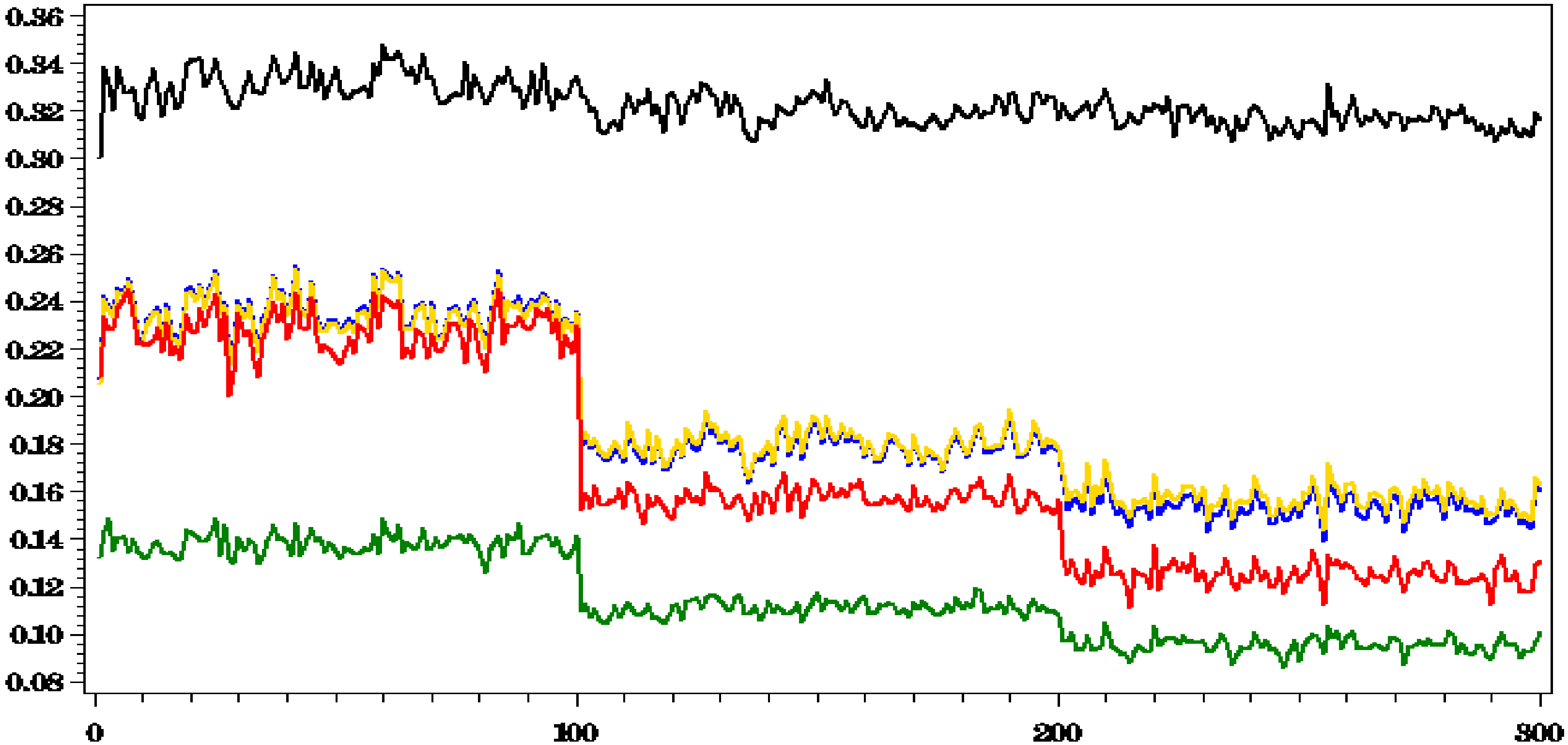
<u>Biases</u>: $\hat{\bar{Y}}_i^{ign}$ - black, $\hat{\bar{Y}}_i^{H,MCAR}$ - **gold,** $\hat{\bar{Y}}_i^{H,MAR}$ **- blue,**

$\hat{\bar{Y}}_i^{H,new}$ **- red,** $\hat{\bar{Y}}_i^{new}$ **- green**



RMSE's: $\hat{\bar{Y}}_i^{ign}$ - black, $\hat{\bar{Y}}_i^{H,MCAR}$ - gold, $\hat{\bar{Y}}_i^{H,MAR}$ - blue,

$\hat{\bar{Y}}_i^{H,new}$ - red, $\hat{\bar{Y}}_i^{new}$ - green



THANKS !!! (Sverchkov.Michael@bls.gov)