

VALIDATION OF MULTICLASS URBAN TRAVEL FORECASTING MODELS COMBINING ORIGIN-DESTINATION, MODE, AND ROUTE CHOICES*

David Boyce

Department of Civil and Materials Engineering, University of Illinois at Chicago, IL 60607, U.S.A. E-mail: dboyce@uic.edu

Hillel Bar-Gera

Department of Industrial Engineering and Management, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. E-mail: bargera@bgumail.bgu.ac.il

ABSTRACT. The formulation, estimation, and validation of combined models for making detailed urban travel forecasts are described. These models combine origin-destination, mode, and auto route choices into a consistent forecasting method for multiple user classes for the Chicago Region. Household Travel Survey and Census Transportation Planning Package data for 1990, respectively, are used to estimate and validate the model.

1. INTRODUCTION

Methods for forecasting urban travel have been at the forefront of regional science and related fields since the 1950s. Innovations and implementation of these methods have proceeded on two distinct fronts, academic research and professional practice. On the academic side, the formulation of the theoretical model of origin-destination demand and user-optimal route

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choice by Beckmann, McGuire, and Winsten (1956) was a truly remarkable innovation for that time. Curiously, this breakthrough was not known to transportation professionals, who were attempting to solve this same problem in early urban transportation studies. Instead they defined and implemented a sequential, or four-step, procedure for travel forecasting that has become the standard paradigm.

The motivation for this paper relates to fundamental research questions pertaining to the formulation and estimation of models that integrate, or combine, travel choices pertaining to origin-destination (O-D), mode, and auto route in a congested urban transportation system. Boyce (2002) has separately addressed the implications of these findings for professional transportation planning practice, including an historical review of both practice and research. Moreover, Boyce and Bar-Gera (2004) have reviewed multiclass combined models and compared four recent model implementations.

Following the original model formulation by Beckmann, McGuire, and Winsten (1956), related formulations and solution algorithms were proposed by Murchland (1970), Evans (1973, 1976), Florian, Nguyen, and Ferland (1975), Boyce et al. (1983), Safwat and Magnanti (1988), Schittenhelm (1990), Fernández et al. (1994), Lundgren and Patriksson (1998), and Abrahamsson and Lundqvist (1999). Each of these proposals concerned a single-class model, in the sense that all travelers by purpose or socioeconomic group are represented as one homogeneous group. Reviews of some of this research were presented by Boyce, LeBlanc, and Chon (1988), Florian and Hearn (1995), Miller (1997) and Boyce and Daskin (1997).

Travelers' choices do vary by class defined with respect to travel purpose, income, and auto availability, and so forth. Accordingly, more recently research has addressed the formulation, solution and implementation of multiclass models. Lam and Huang (1992a, 1992b, 1994) were the first to describe an optimization formulation for the multiclass version, which was implemented for Hong Kong. De Cea and Fernandez (2001) proposed a multiclass combined model formulated as a variational inequality, devised a solution algorithm based on Evans's partial linearization algorithm and described a highly detailed application of this model, ESTRAUS, to Santiago, Chile. Florian, Wu and He (2002) proposed a variant of ESTRAUS intended to be more efficient computationally.

Boyce and his collaborators have implemented two different models for the Chicago region. The first was estimated and implemented with Marshall (Resource Systems Group, 1997) for use in transportation planning activities of public interest organizations. It includes an origin-destination model incorporating both a negative exponential function of generalized cost and the natural logarithm of generalized cost, and was solved heuristically by the method of successive averages. The second is the subject of this paper.

The objective of this research is to estimate and validate a regional travel forecasting model at the level of detail used by transportation planning organizations to forecast road traffic, transit ridership, and vehicle emissions. The model should forecast travel choices on a typical weekday pertaining to

origin-destination and mode, as well as route of auto travel, by purpose in an integrated manner. The solution algorithm should enable planners to solve the model efficiently, and to estimate the values of model parameters in a way that is consistent with the model itself.

The data set available for model estimation is the Household Travel Survey (HHTS) undertaken by the Chicago Area Transportation Study (CATS) during 1989–1991 (Ghislandi, Fijal, and Christopher, 1994). The HHTS contains records from 19,314 households that returned completed questionnaires, representing about 40,000 individuals and 70,000 trips. However, only travel that occurred during the morning peak period, 6:30–8:30 AM, is used to estimate the models.

For the purposes of transportation planning, the 10-county Chicago region is divided into 1,790 zones, as shown in Figure 1. The zones range in size from 1/16 of a square mile in the core of the Central Area to 36 square miles in the outlying counties. External zones surrounding portions of the region represent travel with peripheral regions. Each zone is represented by a zone centroid that is coded into a network representation of the road and transit networks. The road network, consisting of about 39,000 directional links and 13,000 nodes is shown as Figure 2.

Section 2 of the paper describes the formulation of two multiclass models of travel and route choice. Section 3 describes the estimation procedure applied, and presents the estimated values of the model coefficients. In Section 4, the predicted travel choices are compared with an independent data set. Conclusions and suggestions for further research conclude the paper.

2. MODEL FORMULATION

Two multiclass models of origin-destination (O–D), mode, and route choice were formulated, estimated, and validated. The first model corresponds to a simultaneous O-D and mode choice model with a single exponential function of generalized travel cost. The second separates O-D choice and mode choice into separate, but interrelated, exponential functions, and may be described as a nested logit model. Both model formulations are constrained with regard to origins and destinations by class. This terminology was suggested by Abrahamsson and Lundqvist (1999).

In this section, following a statement of the model assumptions, an equivalent optimization problem is formulated, and its optimality conditions are related to standard choice models. Additional constraints are then added to convert this model into the nested logit formulation and its optimality conditions are presented.

Assumptions

Consider travel choices during a typical morning weekday peak period. Travelers are grouped according to their travel purposes into classes

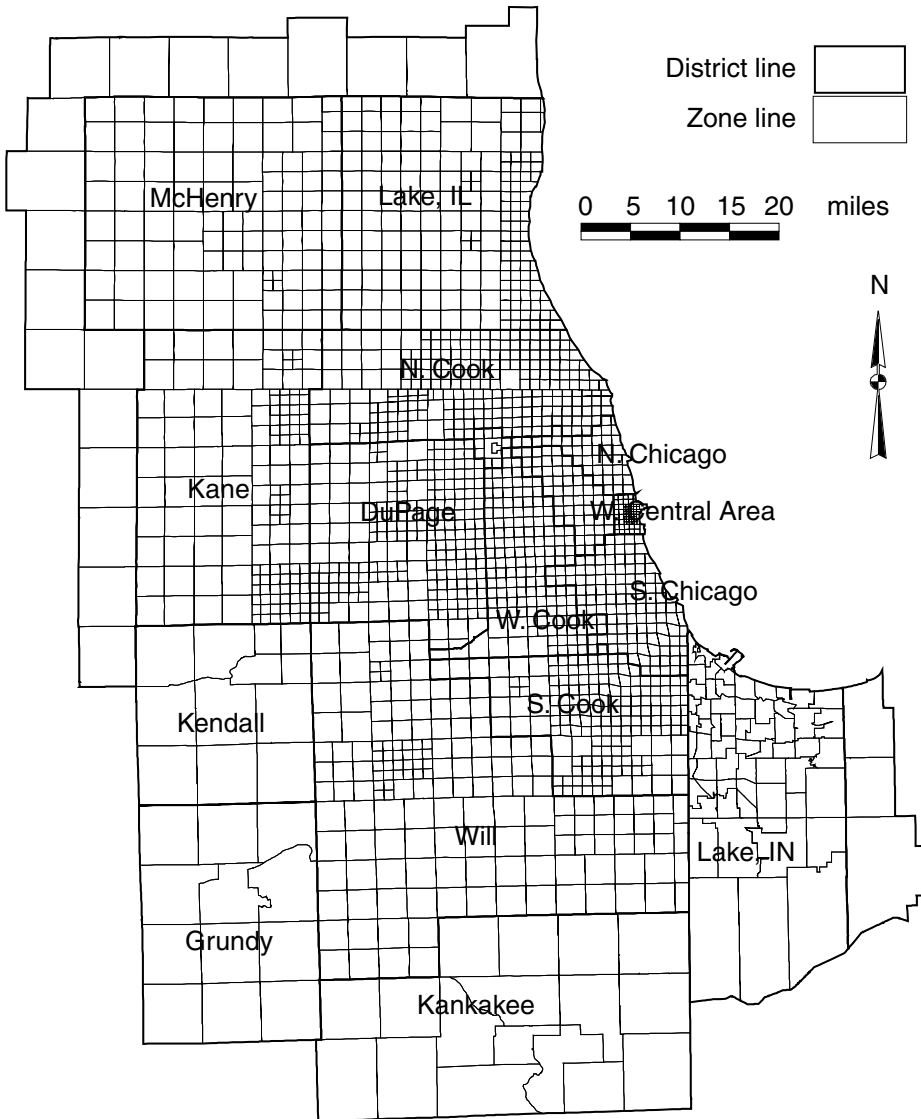


FIGURE 1: Chicago Regional Zone System and Districts.

designated by l . Travelers are represented as constant flows from origins to destinations by mode and route over the peak period in persons per hour. Person flow from origin zone p to destination zone q by mode m for class l is d_{pqm}^l . The total flows of travelers of class l leaving zone p and entering zone q are O_p^l and D_q^l , respectively.

Two modes, auto au and transit tr , are considered to operate over independent networks, so that the generalized costs associated with each mode are

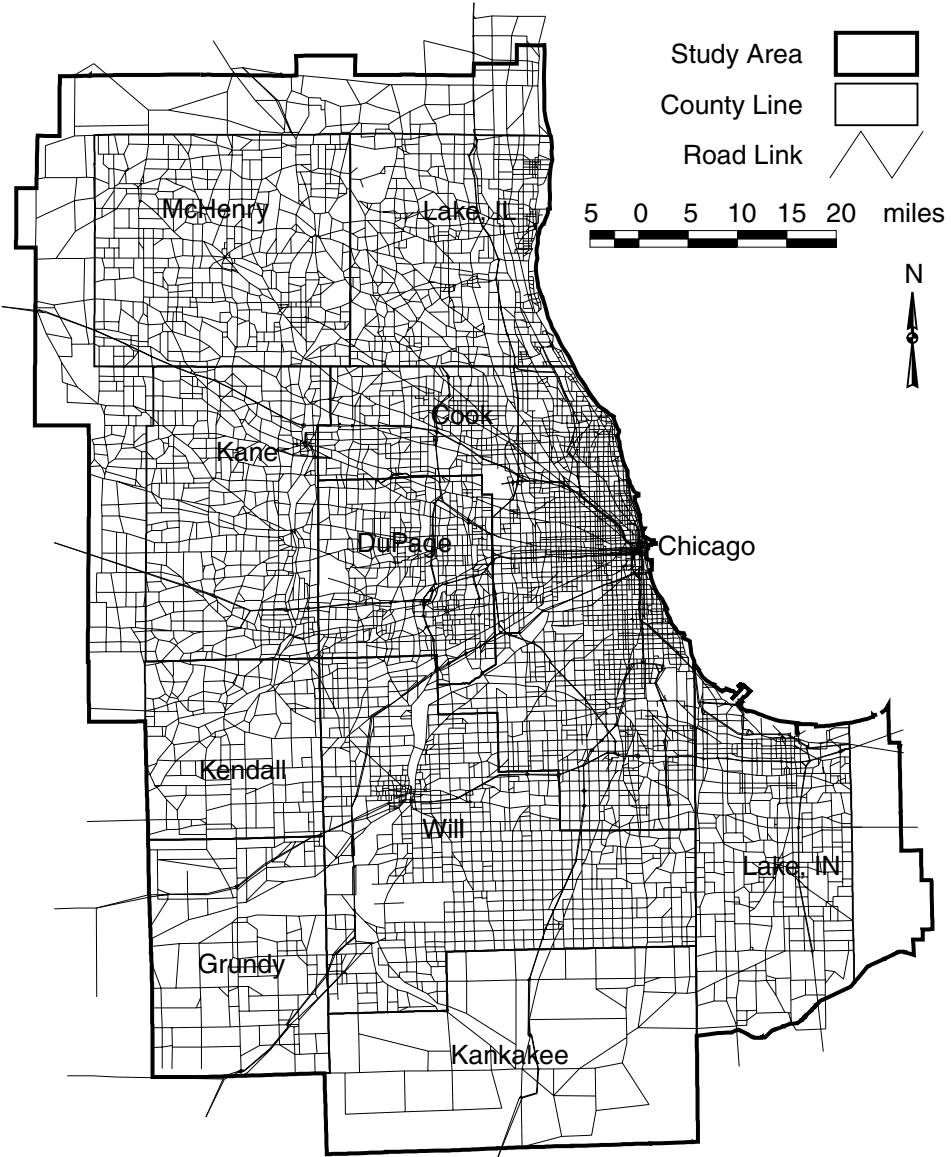


FIGURE 2: Chicago Regional Road Network.

separate. The costs of the transit mode are fixed and given by a timetable and fare schedule. The road network consists of sets of nodes N and links A ; the region is divided into a set of zones Z , with each zone represented by a centroid node. The flow of autos belonging to class l from zone p to zone q on route r of the road network is h_r^l , $r \in R_{pq}$, the set of routes connecting zone p to zone q .

The relation between person flows and auto flows on the road network is represented by a regionwide, class-specific vehicle occupancy factor v^l (persons per vehicle). In addition to autos, fixed truck flows K_{pq} from zone p to zone q result in truck route flows h_r^k in auto equivalent units.

The generalized travel costs are linear, weighted functions of in-vehicle and out-of-vehicle travel time and monetary costs, as well as over-the-route distance in the case of autos. The in-vehicle travel time on each link of the road network is an increasing function of the link's own total flow. Out-of-vehicle times on the road network are fixed access and egress times associated with origins and destinations. Exogenous monetary costs on the road network represent link tolls and parking fees at destinations. The vehicle operating cost of a link is assumed to be an implicit linear function of link travel time (minutes) and link length (miles). No assumption is made concerning the relation of operating cost to link travel time or length, or to vehicle occupancy; instead, time and length are assumed to be variables in the generalized cost function that affect both the disutility of personal travel and the associated auto operating cost. The joint effects of these variables are represented by the estimated coefficients. Transit monetary cost is the transit fare. The variable definitions related to these costs are as follows:

$t_a(f_a)$ = in-vehicle travel time by auto on road link a , a function of the total vehicle flow f_a (minutes)

k_a = vehicle toll per auto equivalent unit on link a or parking fee at the terminal link (cents)

s_a = length of link a (miles)

$\delta_r^a = 1$ if link a belongs to route r , and 0 otherwise

$w_{pq,au}$ = out-of-vehicle travel time by auto for travel from zone p to zone q (minutes)

$t_{pq,tr}$ = in-vehicle travel time by transit from zone p to zone q (minutes)

$k_{pq,tr}$ = transit fare from zone p to zone q (cents)

$w_{pq,tr}$ = out-of-vehicle travel time by transit from zone p to zone q (minutes)

In making their origin-destination-mode choices, travelers are assumed to minimize the generalized cost of travel (disutility), subject to a dispersion of choices to higher cost alternatives, which seeks to account for variables and other factors not included in the model. This dispersion is represented by the entropy function, a measure of dispersion of a frequency distribution, and its associated cost sensitivity coefficient. Choice of route on the road network is assumed to be strictly cost-minimizing with perfect information concerning the generalized cost of the route. This assumption corresponds to the first principle of Wardrop (1952), generally known as user-optimal route choice.

Simultaneous Model Formulation

This simultaneous travel choice problem may be formulated as follows:

(CM-S)

$$\begin{aligned} \text{Min } T(\mathbf{d}, \mathbf{h}) \equiv & \sum_a \int_0^{f_a} t_a(x) dx + \sum_{la} (\gamma_2^l f_a^l k_a + \gamma_4^l f_a^l s_a) + \sum_{lpq} \frac{\gamma_3^l}{v^l} d_{pq,au}^l w_{pq,au} \\ & + \sum_{lpq} \left[\frac{d_{pq,tr}^l}{v^l} (\gamma_5^l t_{pq,tr} + \gamma_6^l k_{pq,tr} + \gamma_7^l w_{pq,tr} + \gamma_8^l) \right] \\ & + \sum_{lpqm} \frac{1}{\mu^l v^l} d_{pqm}^l (\ln d_{pqm}^l - 1) \end{aligned}$$

subject to

$$\begin{aligned} \sum_{r \in R_{pq}} h_r^l &= \frac{d_{pq,au}^l}{v^l}, p, q \in Z; l \in L \\ \sum_{r \in R_{pq}} h_r^k &= K_{pq}, p, q \in Z \\ \sum_{qm} d_{pqm}^l &= O_p^l, p \in Z; l \in L \\ \sum_{pm} d_{pqm}^l &= D_q^l, q \in Z; l \in L \\ h_r^l &\geq 0, r \in R_{pq}, p, q \in Z; l \in L \\ h_r^k &\geq 0, r \in R_{pq}, p, q \in Z \end{aligned}$$

where

$$f_a \equiv \sum_l f_a^l + f_a^k \equiv \sum_{lr} h_r^l \delta_r^a + \sum_r h_r^k \delta_r^a, a \in A$$

Notation used in the above formulation not previously defined is

- $\gamma_1^l = 1$, the coefficient associated with auto in-vehicle travel time for class l
- γ_2^l = the coefficient associated with auto monetary cost for class l
- γ_3^l = the coefficient associated with auto out-of-vehicle travel time for class l
- γ_4^l = the coefficient associated with auto travel distance for class l
- γ_5^l = the coefficient associated with transit in-vehicle travel time for class l
- γ_6^l = the coefficient associated with transit fare for class l
- γ_7^l = the coefficient associated with transit out-of-vehicle travel time for class l
- γ_8^l = the coefficient associated with transit bias for class l
- μ^l = the cost sensitivity parameter for class l

Next, we consider the analysis of the Lagrangian function for this problem.

$$L(\mathbf{d}, \mathbf{h}) = T(\mathbf{d}, \mathbf{h}) - \sum_{lpq} u_{pq}^l \left(\sum_{r \in R_{pq}} h_r^l - \frac{d_{pq,au}^l}{v^l} \right) - \sum_{pq} u_{pq}^k \left(\sum_{r \in R_{pq}} h_r^k - K_{pq} \right) - \sum_{lp} \alpha_p^l \left(\sum_{qm} d_{pqm}^l - O_p^l \right) - \sum_{lq} \beta_q^l \left(\sum_{pm} d_{pqm}^l - D_q^l \right)$$

where $u_{pq}^l, u_{pq}^k, \alpha_p^l, \beta_q^l$ are Lagrange multipliers associated with the corresponding constraints. Taking partial derivatives with respect to the auto route flows, we obtain the following optimality conditions

$$\sum_a t_a(f_a) \delta_r^a + \gamma_2^l \sum_a k_a \delta_r^a + \gamma_4^l \sum_a s_a \delta_r^a - u_{pq}^l \geq 0, pq \in Z; l \in L$$

$$h_r^l \left[\sum_a t_a(f_a) \delta_r^a + \gamma_2^l \sum_a k_a \delta_r^a + \gamma_4^l \sum_a s_a \delta_r^a - u_{pq}^l \right] = 0, pq \in Z; l \in L$$

Similar optimality conditions may be obtained for truck route flows.

Define the auto generalized cost of route r for class l as

$$c_r^l \equiv \left[\sum_a t_a(f_a) \delta_r^a + \gamma_2^l \sum_a k_a \delta_r^a + \gamma_4^l \sum_a s_a \delta_r^a \right]$$

Then, for $h_r^l > 0, r \in R_{pq}, c_r^l = u_{pq}^l$. That is, for all routes from p to q with positive flow of class l , the auto generalized route travel costs are equal. Moreover, for $h_r^l = 0, r \in R_{pq}, c_r^l \geq u_{pq}^l$. Therefore, for routes from p to q with zero flow in class l , the auto generalized travel cost is not less than the cost of routes with flows. These optimality conditions correspond to the first principle of Wardrop (1952).

The units of auto generalized cost are road in-vehicle minutes. The units of the auto-related coefficients are road in-vehicle minutes per unit of the associated variables (cents, out-of-vehicle minutes, miles, etc.). The coefficients for auto monetary cost implicitly include the auto occupancy; however, no explicit assumption is made about how the monetary cost is shared among the occupants. Likewise, the coefficient for length of travel implicitly represents the disutility of travel and the monetary operating cost.

The coefficient of in-vehicle road travel time in the objective function is set to unity. The road link travel time, which depends on total link flow, applies to all classes: Home-to-Work Travel, Other Travel, and Trucks. This construct results in the Jacobian of the link cost function being diagonal, thereby satisfying the symmetry condition for the integration of the travel time function; see Patriksson (1994, p. 52) and Boyce and Bar-Gera (2001) for further details.

Next, consider the partial derivatives with respect to O-D-mode choice variables.

$$\begin{aligned}
 \text{auto} \quad & \left(\frac{\gamma_3^l}{v^l}\right)w_{pq,au} + \left(\frac{1}{\mu^l v^l}\right) \ln(d_{pq,au}^l) + \left(\frac{u_{pq}^l}{v^l}\right) - \alpha_p^l - \beta_q^l > 0, pq \in Z; l \in L \\
 \text{transit} \quad & \frac{1}{v^l}(\gamma_5^l t_{pq,tr} + \gamma_6^l k_{pq,tr} + \gamma_7^l w_{pq,tr} + \gamma_8^l) + \left(\frac{1}{\mu^l v^l}\right) \ln(d_{pq,tr}^l) \\
 & - \alpha_p^l - \beta_q^l > 0, pq \in Z; l \in L
 \end{aligned}$$

Solving the condition for auto, we obtain

$$d_{pq,au}^l = \exp\left[\mu^l v^l (\alpha_p^l + \beta_q^l) - \mu^l (\gamma_3^l w_{pq,au} + u_{pq}^l)\right]$$

Likewise, solving the conditions for transit, we have

$$d_{pq,tr}^l = \exp\left[\mu^l v^l (\alpha_p^l + \beta_q^l) - \mu^l (\gamma_5^l t_{pq,tr} + \gamma_6^l k_{pq,tr} + \gamma_7^l w_{pq,tr} + \gamma_8^l)\right]$$

Using the following definitions

$$\begin{aligned}
 c_{pq,au}^l &= u_{pq}^l + \gamma_3^l w_{pq,au}^l \\
 c_{pq,tr}^l &= \gamma_5^l t_{pq,tr} + \gamma_6^l k_{pq,tr} + \gamma_7^l w_{pq,tr} + \gamma_8^l \\
 A_p^l &= \exp(\mu^l v^l \alpha_p^l) / O_p^l \\
 B_q^l &= \exp(\mu^l v^l \beta_q^l) / D_q^l
 \end{aligned}$$

the travel choice functions may be rewritten as

$$\begin{aligned}
 d_{pq,au}^l &= A_p^l O_p^l B_q^l D_q^l \exp(-\mu^l c_{pq,au}^l) \\
 d_{pq,tr}^l &= A_p^l O_p^l B_q^l D_q^l \exp(-\mu^l c_{pq,tr}^l)
 \end{aligned}$$

The Lagrange multipliers α_p^l and β_q^l , and hence A_p^l and B_q^l , are determined as

$$\begin{aligned}
 1/A_p^l &= \sum_q B_q^l D_q^l \left[\exp(-\mu^l c_{pq,au}^l) + \exp(-\mu^l c_{pq,tr}^l) \right] \\
 1/B_q^l &= \sum_p A_p^l O_p^l \left[\exp(-\mu^l c_{pq,au}^l) + \exp(-\mu^l c_{pq,tr}^l) \right]
 \end{aligned}$$

The following comments may be helpful in understanding the formulation and derivation of the optimality conditions:

- (i) Several terms of the artificial objective function $T(\mathbf{d}, \mathbf{h})$ are divided by the vehicle occupancy of the class. This counterintuitive treatment is necessary to obtain consistent optimality conditions, as shown above.

- (ii) Auto O-D and route choices depend on generalized costs expressed in units of in-vehicle auto travel time, and consist of terms related to link tolls, parking fees and link length, as well as out-of-vehicle auto travel time.
- (iii) Transit O-D choices also depend on generalized costs expressed in units of in-vehicle auto travel time, and consist of in-vehicle transit travel time, out-of-vehicle transit travel time, transit fares and a transit bias term.

Nested Model Formulation

The solution of the above model determines both O-D choice and mode choice simultaneously with a single exponential function of generalized travel costs. To obtain a more flexible model, we consider a formulation with an additional dispersion term in the objective function related to mode choice. In this nested combined model formulation, d_{pq}^l is the total flow of travelers from zone p to zone q in class l , and d_{pqm}^l is the flow from p to q by mode m in class l . This optimization problem is formulated as

(CM-N)

$$\begin{aligned} \text{Min } T(\mathbf{d}, \mathbf{h}) \equiv & \sum_a \int_0^{f_a} t_a(f_a) + \sum_{la} (\gamma_2^l f_a^l k_a + \gamma_4^l f_a^l s_a) + \sum_{lpq} \frac{\gamma_3^l}{v^l} d_{pq,au}^l w_{pq,au} \\ & + \sum_{lpq} \left[\frac{d_{pq,tr}^l}{v^l} (\gamma_5^l t_{pq,tr} + \gamma_6^l k_{pq,tr} + \gamma_7^l w_{pq,tr} + \gamma_8) \right] \\ & + \sum_{lpq} \frac{1}{\eta^l v^l} d_{pq}^l (\ln d_{pq}^l - 1) + \sum_{lpqm} \frac{1}{\mu^l v^l} d_{pqm}^l \left[\ln \left(\frac{d_{pqm}^l}{d_{pq}^l} \right) - 1 \right] \end{aligned}$$

subject to

$$\sum_{r \in R_{pq}} h_r^l = \frac{d_{pq,au}^l}{v^l}, p, q \in Z; l \in L$$

$$\sum_{r \in R_{pq}} h_r^k = K_{pq}, p, q \in Z$$

$$\sum_m d_{pqm}^l = d_{pq}^l, p, q \in Z; l \in L$$

$$\sum_q d_{pq}^l = O_p^l, p \in Z; l \in L$$

$$\sum_p d_{pq}^l = D_q^l, q \in Z; l \in L$$

$$h_r^l \geq 0, r \in R_{pq}, p, q \in Z; l \in L$$

$$h_r^k \geq 0, r \in R_{pq}, p, q \in Z$$

where

$$f_a \equiv \sum_l f_a^l + f_a^k \equiv \sum_{lr} h_r^l \delta_r^a + \sum_r h_r^k \delta_r^a, a \in A$$

Taking partial derivatives of the updated Lagrangian function with respect to d_{pqm}^l , we obtain the following expressions for total O-D flows and modal flows

$$d_{pq}^l = \exp \left[\eta^l v^l \left(\alpha_p^l + \beta_q^l - \frac{1}{\mu^l v^l} \right) - \eta^l v^l \kappa_{pq}^l \right]$$

$$d_{pqm}^l = d_{pq}^l \exp \left(\mu^l v^l \kappa_{pq}^l - \mu^l c_{pqm}^l \right)$$

where κ_{pq}^l is the Lagrange multiplier associated with the conservation of flow constraint relating total O-D flow to O-D flow by mode for class l . By summing the equation for d_{pqm}^l with respect to mode, and applying the conservation of flow constraint, the following expression for κ_{pq}^l may be obtained

$$\kappa_{pq}^l = -\frac{1}{\mu^l v^l} \ln \left[\sum_m \exp \left(-\mu^l c_{pqm}^l \right) \right]$$

Let $\exp \left(-\mu^l \tilde{c}_{pq}^l \right) \equiv \sum_m \exp \left(-\mu^l c_{pqm}^l \right)$.

Thus, $\tilde{c}_{pq}^l = -\frac{1}{\mu^l} \ln \left[\sum_m \exp \left(-\mu^l c_{pqm}^l \right) \right]$, $\kappa_{pq}^l = \frac{1}{v^l} \tilde{c}_{pq}^l$, and

$$d_{pq}^l = \exp \left[\eta^l v^l \left(\alpha_p^l + \beta_q^l - \frac{1}{\mu^l v^l} \right) - \eta^l \tilde{c}_{pq}^l \right] = A_p^l O_p^l B_q^l D_q^l \exp \left(-\eta^l \tilde{c}_{pq}^l \right)$$

$$d_{pqm}^l = d_{pq}^l \frac{\exp \left(-\mu^l c_{pqm}^l \right)}{\sum_m \exp \left(-\mu^l c_{pqm}^l \right)}$$

Therefore

$$d_{pqm}^l = A_p^l O_p^l B_q^l D_q^l \exp \left(-\eta^l \tilde{c}_{pq}^l \right) \frac{\exp \left(-\mu^l c_{pqm}^l \right)}{\sum_m \exp \left(-\mu^l c_{pqm}^l \right)}$$

which is a O-D-mode travel choice model of the nested logit type with endogenous generalized auto travel costs. The above model is one of two nested model formulations, and is sometimes referred to as the traditional model. Another form is the so-called reverse model, which may be stated as

$$d_{pqm}^l = O_p^l \frac{\exp(-\mu^l c_{pm}^{*l})}{\sum_m \exp(-\mu^l c_{pm}^{*l})} \frac{B_q^l D_q^l \exp(-\eta^l c_{pqm}^l)}{\sum_q B_q^l D_q^l \exp(-\eta^l c_{pqm}^l)}$$

where

$$c_{pm}^l = -\frac{1}{\eta^l} \ln \left[\sum_q B_q^l D_q^l \exp(-\eta^l c_{pqm}^l) \right]$$

and c_{pqm}^l is defined as above.

Only the traditional nested model was formulated and estimated in the research reported here. We return to the implications of this model formulation following the presentation of the estimated coefficients. For additional details about the traditional and reverse nested models, see Abrahamsson and Lundqvist (1999). The simultaneous and nested logit models may be solved by a generalization of the partial linearization algorithm proposed by Evans (1976); see Boyce and Daskin (1997) for details.

3. ESTIMATION OF MODEL PARAMETERS

The formulated models have seven coefficients per class plus one or two cost sensitivity parameters, depending on whether the simultaneous or nested case is selected. Other parameters pertaining to auto occupancy by class and the link travel time function are exogenous. In the model implementation described here, two classes are considered: Home-to-Work Travel and Other Travel, plus Trucks. Hence, in the simultaneous model the total number of unknown parameters is 16. In the estimation of the nested model, the generalized cost coefficients were assumed to remain unchanged from the simultaneous model, so only the four cost sensitivity parameters were estimated. In addition the $(2) \times (2) \times (1790) = 7,160$ balancing factors may be considered to be model parameters. These were solved by a standard balancing factor technique.

A generally accepted approach to specifying the parameter estimation problem is the maximum likelihood method (Abrahamsson and Lundqvist, 1999), which corresponds to the following optimization problem

$$\text{Max}_{(\gamma^l, \mu^l)} \ln L^l = \sum_{pqm} N_{pqm}^l \ln \left[P_{pqm}^l (\gamma^l, \mu^l) \right]$$

where

$$P_{pqm}^l (\gamma^l, \mu^l) = \frac{d_{pqm}^l (\gamma^l, \mu^l)}{d^l}$$

$$d^l = \sum_p O_p^l = \sum_q D_q^l$$

N_{pqm}^l = number of trips of class l observed in the travel survey from zone p to zone q by mode m during the period of interest.

The coefficient and parameter values (γ^l, μ^l) pertaining to each class are estimated simultaneously. That is, separate estimation problems are solved for each class, but the values estimated for one class do affect the values in other classes. The ρ^2 goodness-of-fit measure was actually used for each class in the actual estimation process

$$\rho_l^2 = \frac{\ln L_M^l - \ln L_N^l}{\ln L_P^l - \ln L_N^l}, l \in L$$

where

$$\ln L_M^l = \sum_{pqm} N_{pqm}^l \ln \left(\frac{d_{pqm}^l}{d^l} \right), l \in L \quad (\text{model likelihood})$$

$$\ln L_P^l = \sum_{pqm} N_{pqm}^l \ln \left(\frac{N_{pqm}^l}{N^l} \right), l \in L \quad (\text{perfect likelihood})$$

$$\ln L_N^l = \sum_{pqm} N_{pqm}^l \ln \left(\frac{O_p^l D_q^l P_m^l}{(d^l)^2} \right), l \in L \quad (\text{null likelihood})$$

$$N^l = \sum_{pqm} N_{pqm}^l = \text{total observed flow during the period of interest}$$

$$P_m^l = \text{observed proportion of travel of class } l \text{ by mode } m$$

As noted by Abrahamsson and Lundqvist (1999), as well as Boyce (1985), the maximum likelihood problem for each class is a bilevel optimization problem of the form

$$\begin{aligned} & \text{Max}_{(\gamma^l, \mu^l)} \ln L^l \\ & \text{where } (P_{pqm}^l) \text{ solves (CM)} \end{aligned}$$

CM is a nonlinear constrained optimization problem, which is equivalent to the system of equalities and inequalities that constitute its optimality conditions. A single-level optimization problem can be stated by replacing CM by these optimality conditions and all of the associated equality and inequality constraints; this problem is nonconvex in general. Efforts to solve the problem analytically and computationally were investigated by Tian (1999) using a quasi-Newton approach.

This maximum likelihood problem differs from the one typically encountered in travel choice modeling in at least two respects. First, the total flows leaving and entering each zone in the model are different from the total originating and terminating flows in the survey data. That is

$$O_p^l \neq \sum_{qm} N_{pqm}^l, p \in Z$$

$$D_q^l \neq \sum_{pqm} N_{pqm}^l, q \in Z$$

Thus, the originating and terminating flows in the model are exogenous estimates by CATS based on its Household Travel Survey and other data. This approach is necessary for a detailed model implementation because the HHTS data are much too sparse to use as direct estimates of these flows. Our model implementation is for a detailed zone system with 3.2 million O-D pairs per class. Compared to this number, the HHTS-based trip table is extremely sparse with about 10,000 cells having positive flows during the two-hour morning peak period for Home-to-Work Travel and about 1,000 for Other Travel. By comparison, the validation data set, the Census Transportation Planning Package, has about 131,000 cells with positive flows for Home-to-Work Travel only. In contrast, the predicted auto travel table is a full matrix; the predicted transit travel table has a positive entry for each O-D pair with transit service, including zones with auto access to boarding stations.

Second, the travel times and costs used to compute d_{pqm}^l in the estimation procedure are not based on the travel times and costs reported by respondents to the HHTS. Rather, they are computed from the solution of the model for each set of coefficients. In particular, the auto in-vehicle travel times are computed for each set of coefficients by solving the model.

An experimental procedure, devised by Sacks et al. (1989) and Buck (1994), was applied in an attempt to circumvent the above complexities. Very briefly, the procedure may be described as follows:

- (i) assume a range for each parameter, initially from zero to some reasonable upper limit;
- (ii) draw a random sample of $i = 1, \dots, n$ sets of parameter values from the assumed ranges;
- (iii) solve the travel choice model for each set of parameter values, resulting in n values of $\rho_i^2(i)$ for each class;
- (iv) fit a response surface $\rho_i^2(i) = f(\gamma_i, \mu_i), i = 1 \dots n$ to the set of values for each class;
- (v) using techniques developed by Buck (1994) identify the maximum value $\hat{\rho}_i^2(i)$ together with its associated values of $(\hat{\gamma}, \hat{\mu})$
- (vi) repeat steps (i)–(v) for more restricted ranges of each parameter identified as containing the values corresponding to the optimal ρ_i^2 .

In this application the number of sets of parameter values was 100. The procedure yielded satisfactory results in two to three iterations of the method.

The estimation procedure was applied in the following way. First, the coefficients of the simultaneous model were estimated. The best coefficient and ρ^2 values obtained from the estimation process are shown in Table 1. In considering these ρ^2 values, recall that the null hypothesis is based on exogenous origin and destination flows, and the regional mode share, rather than the null hypothesis that each cell in the trip table has the same value, as is sometimes assumed. Our null hypothesis is more realistic and conservative. Although the ρ^2 values are not especially large, they are sensitive to the model coefficients. The Home-to-Work Travel and Other Travel coefficients are effectively uncorrelated, which simplified the simultaneous estimation process substantially.

The estimated values of the coefficients appear to be reasonable in relation to the units. Because the length of route variable represents the disutility of length of travel and the monetary cost of travel, ratios of the values cannot be interpreted as values of time. For Home-to-Work Travel, the transit bias coefficient is estimated to be zero, whereas for Other Travel, it is 0.09 minutes, a small value compared to the mean generalized travel cost of 18.6 minutes. These bias values bring the regional estimates into agreement with the observed regional mode shares shown in Table 2.

This simultaneous model was validated with the 1990 Census Transportation Planning Package (CTPP) for Home-to-Work Travel (U.S. DOT, 1995). Although the results were considered to be very good, O-D flows between suburban districts separated by 10 to 30 miles were observed to be under-predicted by the model. For this reason, we decided to estimate the traditional

TABLE 1: Estimated Values of Coefficients for Two Multiclass Travel Choice Models

Travel Class	Home-to-Work Travel		Other Travel	
	Auto	Transit	Auto	Transit
Generalized Cost Coefficients (units)				
In-vehicle Travel Time (gc/minute)	1.0	0.25	1.0	0.28
Out-of-vehicle Travel Time (gc/minute)	0.0	0.90	0.47	0.46
Monetary Cost (gc/cents)	0.049	0.084	0.0003	0.052
Length of Route (gc/miles)	0.15	N/A	0.05	N/A
Transit Bias (gc)	N/A	0.00	N/A	0.09
<i>Cost Sensitivity Coefficients</i>				
Simultaneous O-D-Mode		0.14		0.25
Nested Logit				
O-D		0.13		0.28
Mode		0.15		0.20
<i>Goodness-of-Fit ($0 \leq \rho^2 \leq 1$)</i>				
Simultaneous		0.19		0.33
Nested Logit		0.21		0.35

gc: generalized cost of travel in auto in-vehicle minutes

Note: All estimated values are rounded to two significant digits.

TABLE 2: Summary Means for the Simultaneous and Nested Logit Models

Class		Home-to-Work Travel			
Mode		Auto		Transit	
Model	unit	Simultaneous	Nested Logit	Simultaneous	Nested Logit
In-vehicle time	minutes	16.1	17.7	25.5	24.9
Out-of-vehicle time	minutes	2.7	2.7	12.8	13.1
Total travel time	minutes	18.8	20.4	38.3	38.0
Monetary cost	dollars	0.31	0.34	1.46	1.46
Length	miles	10.8	11.5	N/A	N/A
Generalized cost	minutes	20.1	21.3	30.0	30.2
<i>Observed gen. cost</i>	<i>minutes</i>	<i>18.4</i>	<i>18.6</i>	<i>29.7</i>	<i>29.7</i>
Regional share	percent	83.7	83.7	16.3	16.3
<i>Observed reg. share</i>	<i>percent</i>		<i>83.8</i>		<i>16.2</i>
Central Area share	percent	34.8	35.0	65.2	65.0
<i>Observed CA share</i>	<i>percent</i>		<i>36.0</i>		<i>64.0</i>

Class		Other Travel			
Mode		Auto		Transit	
Model	unit	Simultaneous	Nested Logit	Simultaneous	Nested Logit
In-vehicle time	minutes	8.9	8.4	16.3	14.5
Out-of-vehicle time	minutes	2.5	2.6	15.5	14.0
Total travel time	minutes	11.3	10.9	31.9	28.5
Monetary cost	dollars	0.15	0.14	1.32	1.28
Length	miles	5.4	4.9	N/A	N/A
Generalized cost	minutes	11.8	11.2	18.6	17.2
<i>Observed gen. cost</i>	<i>minutes</i>	<i>10.3</i>	<i>10.4</i>	<i>18.9</i>	<i>18.9</i>
Regional share	percent	82.7	83.0	17.3	17.0
<i>Observed reg. share</i>	<i>percent</i>		<i>82.8</i>		<i>17.2</i>
Central Area share	percent	40.8	51.3	59.2	48.7
<i>Observed CA share</i>	<i>percent</i>		<i>41.0</i>		<i>59.0</i>

Italicized rows denote observed values.

Observed generalized costs are different for the Simultaneous and Nested Logit models because of differences in model coefficients that determine these values.

nested model described above. In the estimation of the nested model, the generalized cost coefficients were held fixed, and only the cost sensitivity coefficients were estimated. The ρ^2 values from this process improved only slightly, as also shown in Table 1.

For Home-to-Work Travel the mode cost sensitivity coefficient of the traditional nested model was found to be larger than the O-D cost sensitivity coefficient, suggesting that travelers are slightly more cost minimizing in their mode choices than in their origin-destination choices; see Williams

(1977) for further discussion of this issue. However, the differences are small. For Other Travel, the opposite situation was encountered: the mode cost sensitivity coefficient is considerably smaller than the O-D cost sensitivity coefficient. This finding suggests the consideration of the reverse nested model as an alternative hypothesis. Lack of time and resources prevented us from pursuing this hypothesis.

Table 2 compares the two models in terms of mean values at the regional scale for the two travel classes. The table is helpful in visualizing the general variations represented in the model across modes and travel classes: transit travel is longer in time than auto travel; Home-to-Work Travel is longer than Other Travel. The regional share of trips by mode was established in the simultaneous model by the estimation of the bias coefficient, but is free to vary in the nested model. The transit share of travel terminating in the Central Area is a model evaluation measure, and not directly determined by any coefficient.

The response of the predicted travel patterns to the change in model structure and parameters at the regional level is also illustrated by Table 2. For Home-to-Work Travel, auto travel becomes about nine percent longer in time (1.6 minutes), and six percent longer in length (0.7 miles), whereas transit travel remains about the same. Regional mode share remains constant, as does mode share to the CBD. For Other Travel, both auto and transit travel decrease in travel time (0.4 minutes) and length (0.5 miles) because of the increased O-D cost sensitivity, and transit loses mode share slightly. However, auto mode share to the Central Area increases by 25 percent, which is substantially different from the observed level.

4. MODEL VALIDATION

Only the validation of the nested logit model is presented. The validation data set is the 1990 Census Transportation Planning Package (CTPP) (U.S. DOT, 1995). From this data set, origin-destination matrices by the auto and transit modes for Home-to-Work Travel were constructed. For some purposes, this data set was further reduced to the principal six counties that comprise the Chicago metropolitan area. No data is available from the CTPP for Other Travel. The period of travel was restricted to the morning peak period, defined to be 6:30 to 8:30 AM in the CTPP.

The validation proceeded along two lines. First, model estimates and observed travel for O-D zone pairs were aggregated by airline travel distance into one-mile intervals ranging from 1 to 30 miles. Airline distance was chosen as a neutral variable for comparing the model estimates and data. Travel time could also be used; however, auto and transit travel times for a given O-D zone pair are often very different, making the comparison much less meaningful. Intrazonal travel is shown in the figures as the zero distance interval, although intrazonal journey lengths ranging from 0.5 to 3 miles were assumed in the model, depending on the zone size. Second, zones within the six-county

metropolitan area were aggregated into 12 districts shown in Figure 1: four districts in the City of Chicago; three districts in suburban Cook County; and five districts corresponding to the five collar counties. Comparisons of the model and CTPP data were constructed based on these district-to-district flows.

Figures 3–6 present validation results for Home-to-Work Travel. Figure 3 shows the total O-D travel share (auto and transit) for airline distance intervals ranging from 1 to 30 miles, plus intrazonal travel, shown at interval 0. Note the O-D shares are plotted on a logarithmic scale. Although the results are rather aggregated, the correspondence between the model and data shares is clearly quite close. Figure 3 suggests that the model overestimates travel shares for travel in the 4–10 mile range and underestimates shares for the 10–30 mile range. Examination of the shares of intrazonal travel, shown as interval 0, shows that the model poorly estimates this situation; the CTPP shares (0.028) are over twice as large as the model (0.012). These results indicate a missed opportunity to tune the model. The intrazonal travel times and distances are essentially guesses for each zone size. By adjusting these values, the effects of these guesses on the model could have been brought into agreement with the HHTS.

Figure 4 shows the share of Home-to-Work Travel using transit for the model and the CTPP; the regional transit shares are also shown for both data sets. Generally, the correspondence is good for the flows ranging from 2 to 19 miles. Longer flows, which are smaller in magnitude, show considerable scatter. As with the O-D shares, the estimates of the intrazonal transit share is poor, and could have been improved by tuning the intrazonal modal travel times, costs, and distances. Journey times by auto and transit versus airline

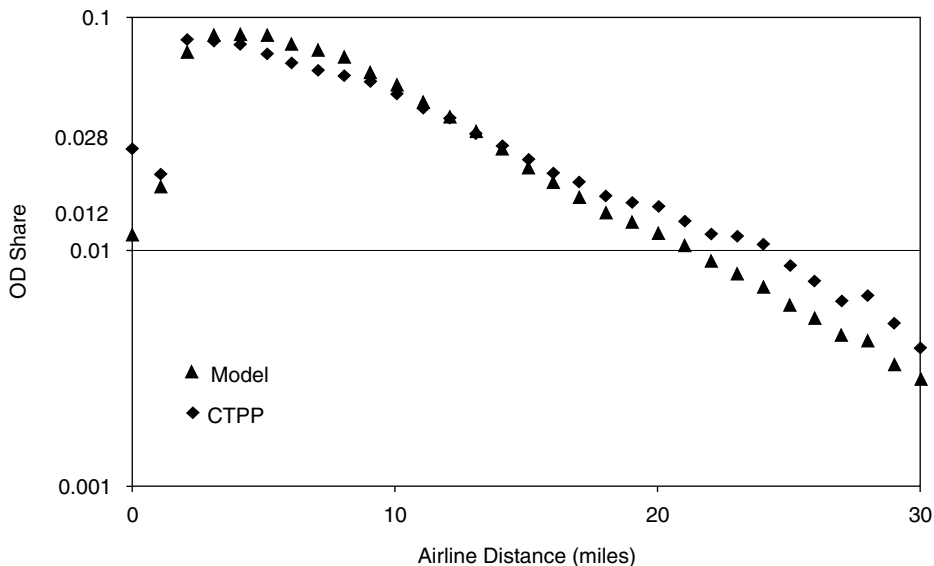


FIGURE 3: Origin-Destination Share of Home-to-Work Travel by Distance.

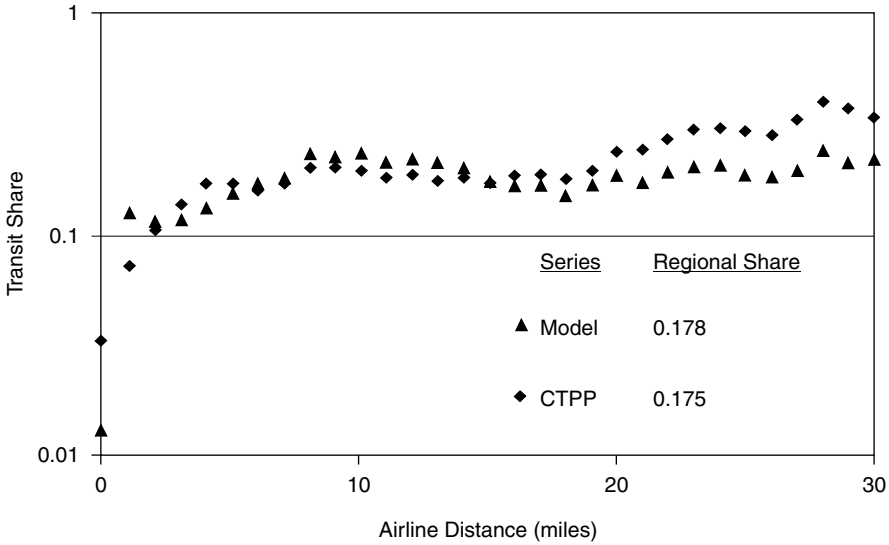


FIGURE 4: Transit Share of Home-to-Work Travel by Distance.

distance are shown in Figures 5 and 6 for the model and the CTPP data set. The results are reasonably similar and relatively constant, and suggest another missed opportunity for tuning the model with regard to access and egress times of both auto and transit travel.

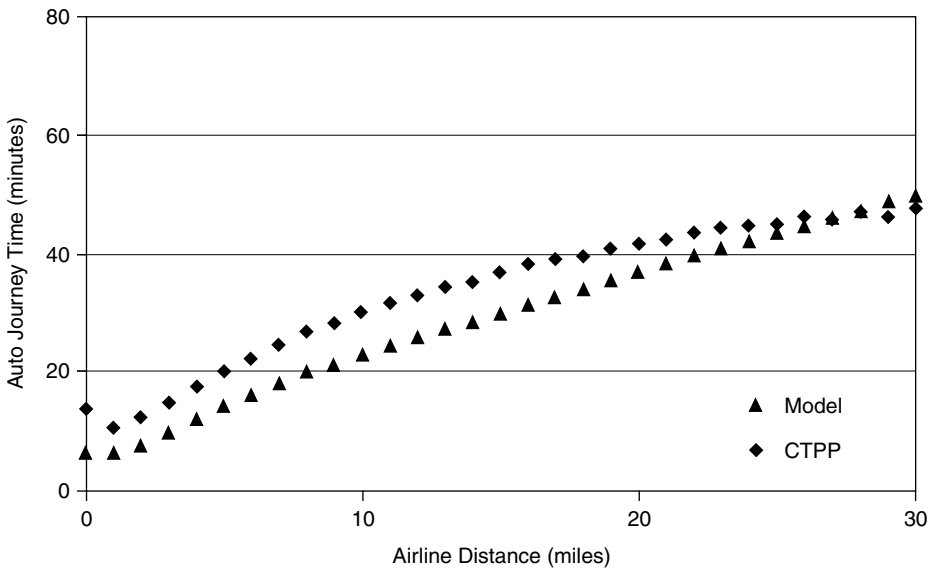


FIGURE 5: Mean Auto Journey Time for Home-to-Work Travel by Distance.

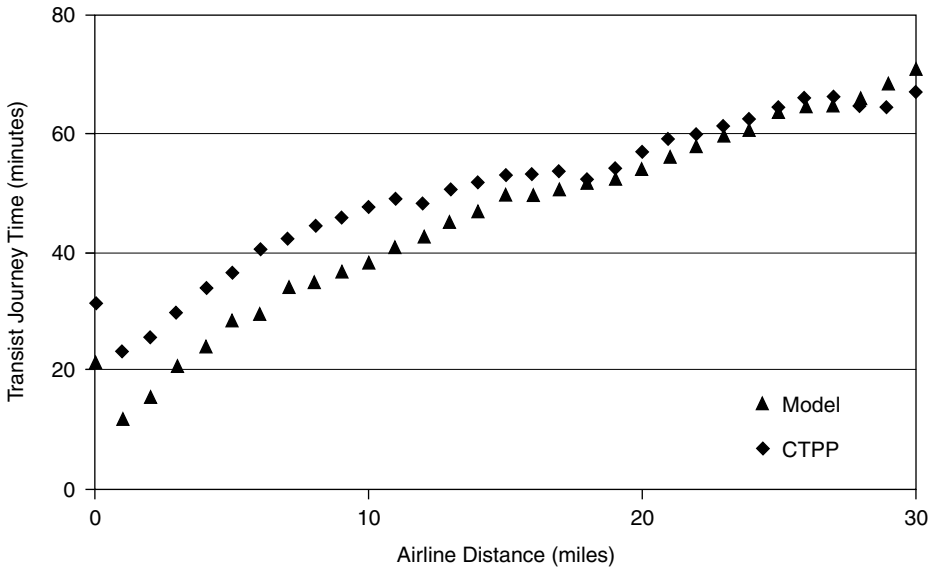


FIGURE 6: Mean Transit Journey Time for Home-to-Work Travel by Distance.

Figure 7 compares the estimated (model) and CTPP O-D shares in a different way. The travel shares were aggregated to 12 districts listed in Figure 7 and shown in Figure 1. The ratio of the share of CTPP travel between each pair of districts to the model share is shown on the vertical axis against the model share on the horizontal axis. The upper diagram shows the O-D pairs by origin district, whereas the identical plot in the lower diagram shows the O-D pairs by destination district. By selecting a point and noting its upper and lower symbol, the origin and destination districts can be identified.

The shape of the plot in Figure 7 clearly shows that the model underestimates flows between districts with small shares estimated by the model, which are the more widely separated pairs. For example, the share of travel from South Cook to North Cook reported by the CTPP is 6.3 times the value of 0.0009 estimated by the model; hence, if this flow were estimated correctly, it would be about 0.006. Likewise, the share from North Cook to South Cook is nearly 40 times the value of 0.00011 estimated by the model. Hence, its value should be more than 0.004. These examples illustrate the general trend that shares between highly separated districts are underestimated. Similar results for auto and transit shares (not shown) provided substantial insights into the goodness-of-fit and weaknesses of the model.

Additional studies should be undertaken for a more complete validation of a model. First, comparisons of predicted link flows with observed link flows ought to be made. Although flows are monitored by time of day on expressways in the Chicago region, these data were not preserved for the 6:30–8:30

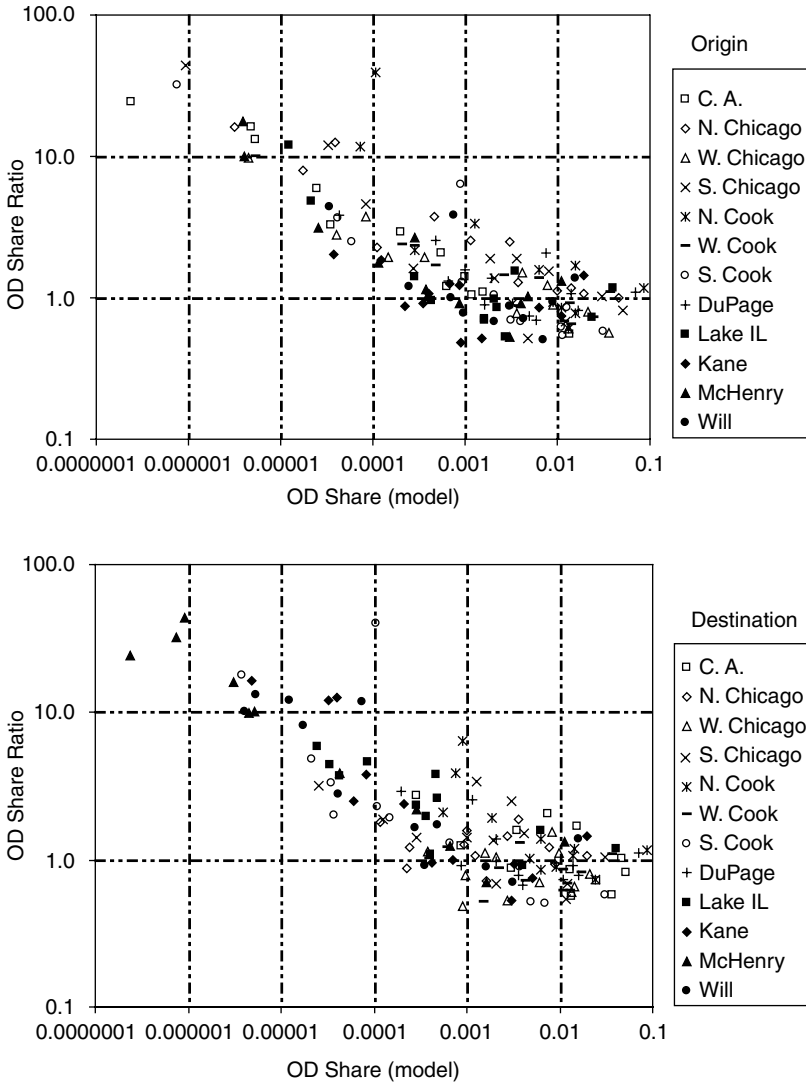


FIGURE 7: Ratio of CTPP OD Shares to Model OD Shares for Home-to-Work Travel.

AM period for a typical weekday in 1990. Of course, flows are not presently monitored by class of traveler.

An application of the model following a substantial change in the road or transit network, or changes in other conditions, would also be useful. Moreover, future comparisons of model predictions with the 2000 CTPP would be desirable.

5. CONCLUSIONS

The research findings summarized in this paper describe the formulation, estimation, and validation of a large-scale, multiclass model of peak period urban travel. The findings demonstrate the feasibility of implementing and using a model that achieves the internal consistency between O-D-mode flows and auto travel times regarded as important to travel forecasting. In other words, all feedback relationships are rigorously represented in the formulation and implemented in the model solution. Moreover, the estimated coefficients are internally consistent with the model structure and the endogenous predicted travel times over the road network.

Limitations of time and budget, as well as lack of perfect foresight, tended to reduce the quality of the results below what might have been accomplished otherwise. Nevertheless, the model was estimated and validated in a completely new way from the viewpoint of urban transportation research and practice. Even so, more detailed model testing and implementation studies remain to be accomplished before this model can be regarded as ready for use in practice. Alternative model functional forms should be investigated, especially the more general deterrence function that combines the negative exponential function with the power function. In adopting that function, it appears that the optimization framework used in this research must be discarded. However, recent research findings by Bar-Gera and Boyce (2003) illustrate the feasibility of solving such models directly in a fixed-point framework.

Finally, for such a model formulation and estimation procedure to be applied in professional practice, improved software systems need to be devised. Although much progress has been made in the last decade in software systems for transportation planning, the requirements for this model go well beyond existing systems. Nevertheless, we are optimistic that ongoing improvements will be made available to the professional community in the future.

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