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Technical Report Number 89
August, 1998

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Abstract

In order to analyze random set processes, it is necessary to have simple statistics that can be used to describe their outcomes. The cumulants and several other parameters can be used for this purpose, but their estimates can be excessively variable if the most straightforward estimators are used. Through exploitation of similarities between this estimation problem and a similar one for a point process statistic, two modifications are suggested. Clear analytical results concerning the effects of these modifications were found through use of a specialized asymptotic regime, whose form was related to certain geometrical aspects of the modifications. Simulation results established that the modifications were highly effective at reducing estimator standard deviations for several Boolean models. The results suggested that the reductions in variance resulted from a balanced use of information in estimation of first and second moments, through eliminating the use of observations that were not used in second moment estimation.

Keywords: random set, germ-grain models, cumulants, edge effects

AMS Classification Numbers: 62M30 (primary), 60D05 (secondary)

1 Introduction

Random sets may be simple to simulate, yet their outcomes are difficult to describe. The law of the process is not accessible, and the moments of the process may be difficult to relate to the interesting features of the geometry of an outcome. Even so, standardized moments can be useful in determining the structure of dependence in a random set, but the most intuitive estimators of these parameters possess greater sampling variances than certain modified estimators. In a seeming paradox, these estimators may make use of much less information than the most intuitive estimators. Careful examination of the structure of the modified estimators explains this paradox away by demonstrating that these estimators make much better use of the available information, and disregard information that proves to be extraneous to the estimation problem in question.

2 Notation and Terminology

In the most general case, a realization of a random set Φ in $\mathbb{R}^d$ can be represented by the indicator functions defined at all $x \in \mathbb{R}^d$ by

$$I_\Phi(x) = \begin{cases} 1 & \text{if} \quad x \in \Phi \\ 0 & \text{if} \quad x \notin \Phi. \end{cases}$$

Matheron [1] demonstrated the existence of a probability measure that allows such processes to be considered as stochastic processes in the conventional sense, but the laws of such processes have no convenient and succinct analytical expressions. Instead, random sets in common use are defined in terms of the algorithm that produces them and so are generally referred to as *models*. Classic examples of such models are the Boolean Model [2] and the Dead Leaves Model [3], both of which belong to the more general category of germ-grain models [4]. Much of this paper is concerned with these models, which use the points in the realization of a point process (know as *germs*) as locations at which geometric objects of possibly random shape, size and orientation (known as *grains*) can be attached.

Even if it is not possible to describe the laws of random sets easily, the moments of these processes can be straightforwardly described and estimated. Since a random set process is a collection of random indicator functions indexed by $\mathbb{R}^d$, its first moment at any point $x \in \mathbb{R}^d$ will be

$$M_1(x) = \mathrm{E}[I_\Phi(x)] = \Pr[x \in \Phi].$$

Similarly, the second moment of the function can be defined at any two points $x, x+r \in \mathbb{R}^d$ by

$$M_2(x, x+r) = \mathrm{E}[I_\Phi(x) I_\Phi(x+r)] = \Pr[x, x+r \in \Phi]$$

and in general the k^{th} moment at points $x, x+r_1, \ldots, x+r_{k-1} \in \mathbb{R}^d$ is defined by

$$M_k(x, x+r_1, \ldots, x+r_{k-1}) = \Pr[x, x+r_1, \ldots, x+r_{k-1} \in \Phi].$$

These moments will be meaningful only if the random set has positive measure. Although a point process is also a random set according to the criteria given, analysis of this type of random set requires different analytical methods which are described in Daley and Vere-Jones [5], Diggle [6], Ripley [7, 8], Stoyan et al. [4], and Cressie [2]. While these methods are quite distinct from those given here, some ideas can carry over from one type of random set to the other, as will be shown in the development of the modified estimators.

In order to use random set models in a statistical context, some notion of replication is required. One way in which replication can occur is through the random set process being *stationary*. Abstractly, a stationary

random set is one whose probability measure is invariant under translations acting on the index set, but for the results here this invariance might only be required of the first few moment functions. For example, if a process were k^{th}-moment stationary, then for any vectors $x_1, \ldots, x_k, y \in \mathbb{R}^d$,

$$M_k(x_1, \ldots, x_k) = M_k(x_1 + y, \ldots, x_k + y).$$

For the first moments of all stationary random sets, this translation invariance property implies that the point at which it is considered is irrelevant. For this reason, that point is arbitrarily be chosen to be zero, and so $M_1(x)$ can replaced by the *reduced first moment*, which is defined as

$$m_1 = M_1(0) = Pr[0 \in \Phi].$$

By fixing at 0 one of the the points at which the k^{th} moment is taken, the *reduced* k^{th} *moment* is defined as

$$m_k(x_1, \ldots, x_{k-1}) = M_k(0, x_1, \ldots, x_{k-1}) = Pr[0, x_1, \ldots, x_{k-1} \in \Phi].$$

This is not the standard notation [4], which designates the first reduced moment as p and the second reduced moment as $C(r)$. There is no standard definition for higher moments, and the conventional notation introduces unnecessary obscurity when third and fourth moments are considered.

Aside from allowing simpler definitions for moments, reduced moment estimates for stationary processes are easily estimable. The first moment is estimated by the fraction of a window of observation A covered by the process:

$$\widehat{V}_1 = \frac{\nu_d(\Phi \cap A)}{\nu_d(A)},$$

where ν_d is Lebesgue measure on $\mathbb{R}^d$. This estimate is an idealization, but avoids complicating the analysis by introducing discretization.

In order to have the estimator $\widehat{V}_1$ converge to its desired limit m_1 by a law of large numbers, it is also necessary to assume that the random set is ergodic. A related assumption, called the mixing condition, is required to resolve some difficult mathematical issues, but ergodicity as formally defined is not directly required in any further discussion of estimator covariances.

For higher order moment estimation, it is necessary to define the translate of a set $A \subset \mathbb{R}^d$ by a vector $r \in \mathbb{R}^d$ as

$$A_r = \{x : x - r \in A\}.$$

The second moment of a stationary random set associated with $r \in \mathbb{R}^d$ can be estimated by looking at all pairs of points $x \in A \cap A_{-r}$ and $x + r \in A \cap A_r$ and then determining the fraction of the points in $A \cap A_{-r}$

for which both x and $x+r$ belong to Φ:

$$\widehat{V}_2(r) = \frac{\nu_d(\Phi \cap \Phi_r \cap A \cap A_r)}{\nu_d(A \cap A_r)}.$$

Estimators for higher moments can be constructed in a similar fashion. For example, for $r, s \in \mathbb{R}^d$

$$\widehat{V}_3(r,s) = \frac{\nu_d\left(\Phi \cap \Phi_r \cap \Phi_s \cap A \cap A_r \cap A_s\right)}{\nu_d\left(A \cap A_r \cap A_s\right)}.$$

If a random set process is stationary and its law is invariant under rotations of its coordinate system, then the process is *isotropic*. For such a process, its second moment $m_2(r)$ depends only on the length of the vector r, not on the orientation of r. In the case of a k^{th} moment for an isotropic process considered for the configuration of points $0, x_1, \ldots, x_{k-1}$, isotropy implies that the moment has the same value if that configuration is subjected to an arbitrary rotation.

3 Measures of Dependence in Random Sets

In the study of random sets, the reduced second moment of a stationary random set process $m_2(r)$ can be used to identify certain patterns present in those sets. In particular:

- The distance at which the dependence structure becomes negligible. In many processes of general interest, it is reasonable to expect that if two points $x, y \in \mathbb{R}^d$ are separated by a great distance, then the events $x \in \Phi$ and $y \in \Phi$ should be independent. For example, in a Boolean model whose grains are circles of diameter μ, the second moment will satisfy

  $$m_2(r) = m_1^2 \quad \text{whenever} \quad |r| > \mu.$$

  In other types of ergodic random set processes, one would expect to find that $m_2(r) \approx m_1^2$ when $|r|$ is very large.

- The presence of repulsion. If the random set tends to produce realizations which have discrete, disconnected components that are forced to be somewhat isolated from each other, then this may result in $m_2(r) < m_1^2$ for some values of r and damped oscillation of the second moment function around the line $m_2(r) = m_1^2$ for larger values of r. Such a process would result if a germ-grain model had grains of fixed size and shape, but germs that were not allowed to overlap.

While these attributes can be seen directly in the second moment, it is also possible to standardize the second moment so that the condition of independence manifests by having the second moment take on a fixed value. For $m_2(r)$, there are three standardizations:

- The second ordinary cumulant, or covariance.

$$\kappa_2(r) = m_2(r) - m_1^2$$

- The second spatial cumulant.

$$\Sigma_2(r) = \frac{m_2(r)}{m_1^2}$$

- The correlation.

$$\rho(r) = \frac{m_2(r)}{m_1(1-m_1)}$$

Of these three, the $\kappa_2(r)$ is of interest both as a classical standardization and as one whose estimators can be most easily analyzed for their asymptotic properties. The ordinary cumulant functions are not as subtle at describing the structure of random sets as are the spatial cumulants [9], and so these are of potentially greater interest in the context of standardized higher moments of random sets. The covariance cannot be generalized to higher order moments, and so will not be discussed in depth.

4 Strategies For Variance Reduction

The estimation of the $\kappa_2(r)$ and $\Sigma_2(r)$ appears to be a straightforward task. Their most intuitive estimators are

$$\widehat{\kappa}_2(r) = \widehat{m}_2(r) - \widehat{V}_1^2$$

and

$$\widehat{\Sigma}_2(r) = \frac{\widehat{m}_2(r)}{\widehat{V}_1^2},$$

both of which would appear to make the best use of the information in the window. Since these estimates would be used primarily to locate the distances at which the dependence structure vanishes, it can be shown that this is not the case. While the exploitation of isotropy presents one method for improving these estimators, it is also possible to exploit a second idea known as balancing that allows further improvements to be made.

4.1 Modification Through Isotropy

The most obvious method for reducing variance in an estimator is to use as much available information as possible. If the random set is isotropic, then the estimator of $m_2(r)$ that best exploits the extra information that isotropy can provide is

$$\widehat{m}_2^I(r) = \frac{\int_{C_d(R)} \nu_d(\Phi \cap \Phi_r \cap A \cap A_r)dr}{\int_{C_d(R)} \nu_d(A \cap A_r)dr}$$

where $C_d(R)$ is the sphere of radius R in $\mathbb{R}^d$. This estimator is superior to one which averages $\widehat{m}_2(r)$ over all $r \in C_d(R)$, since unless A is a sphere in $\mathbb{R}^d$, the latter estimator will weight estimates equally, unaffected by the changes in the size $A \cap A_r$ that occur with changes in the direction of r.

4.2 Modification Through Balancing

A second strategy often used in the reduction of estimator variance involves adding some statistic $\widehat{H}$ to $\widehat{m}_2(r)$, chosen so that $E[\widehat{H}] \approx 0$ and $cov[\widehat{H}, \widehat{m}_2(r)] < 0$ This strategy was used to modify a point process estimator which had a similar form to that of $\widehat{\Sigma}_2(r)$, and the structure of that modification suggested the form of a modification for the random set estimator.

4.2.1 Heuristic Argument For The Development of A Balance Modification

Suppose that N is a stationary, ergodic, isotropic point process on $\mathbb{R}^d$. The second moment of a point process indicates the degree to which the points tend to avoid each other or form clusters, but this is not immediately reflected in the formal definition of the second moment, which for $\forall A, B \subset \mathbb{R}^d$ is $E[N(A)N(B)]$. Ripley [10] was able to decompose this formal expression so that the aspects of the second moment related to clustering and repulsion could be described by a function related to the distances between pairs of points, known as the K-function:

$$K(R) = \left(\frac{1}{\lambda}\right) \mathrm{E}\left[\text{ number of points within } R \text{ of } x \mid x \text{ is a point in the process }\right] \tag{1}$$

where λ is the intensity of the process. The K-function is a standardized measure of second moment behaviour, and its simplest estimate has a form very similar to that of $\widehat{\Sigma}_2(r)$. If $\widehat{N}(A)$ is the number of points observed in a window A, then this estimate is:

$$\widehat{K}(R) = \frac{\widehat{T}(R)/\widehat{N}(A)}{\widehat{N}(A)/\nu_d(A)} = \nu_d(A)\left(\frac{\widehat{T}(R)}{\widehat{N}(A)^2}\right) \tag{2}$$

where

$$T(R) = \sum_{x,y \in N \cap A} \phi(x,y) \tag{3}$$

and

$$\phi(x,y) = \begin{cases} 1 & \text{if} \quad 0 < |x-y| \leq R \\ 0 & \text{if} \quad |x-y| > R \quad \text{or} \quad x = y \end{cases}$$

Estimators of functions based on the second moment of a point process are generally U-statistics such as (3), even after they have been modified by the addition of edge corrections [8]. For any estimate of this form, Stein [11] demonstrated that it is possible to use the Hajek Projection Lemma to develop a modification that optimally reduces asymptotic variance. If $X_1, \ldots, X_n$ are i.i.d. and uniformly distributed on a subset A of $\mathbb{R}^d$, then for a given symmetric function $\phi(\cdot,\cdot)$,

$$\text{var}\left(\sum_{i \neq j} \phi\left(X_i, X_j\right) - \sum_{j=1}^{n}\left(g\left(X_j\right) - \frac{1}{\nu_d(A)}\int_A g(x)dx\right)\right)$$

is minimized by

$$g(x) = (n-1)\left(E\left[\phi\left(X_1, X_2\right)|X_1 = x\right] + E\left[\phi\left(X_1, X_2\right)|X_2 = x\right]\right). \tag{4}$$

When a second moment estimator that has been modified by (4) is used in an estimator of $K(r)$, Stein [12] was also able to show that the degree of reduction in the asymptotic variance is quite large.

To construct a similar estimator for random sets, it is necessary to find a parameter whose estimator is of the same structure as (3), but which is an integral with respect to Lebesgue measure rather than counting measure. On a stationary, isotropic random set this parameter is

$$\int_{S_d(R)} m_2(r)dr \tag{5}$$

where $S_d(R)$ is the ball of radius R in $\mathbb{R}^d$, rather than its surface $C_d(R)$. An estimator of (5) which exploits the isotropy is

$$\int_0^R \widehat{m}_2^I(s)ds = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{I_{\Phi\cap A}(x) I_{\Phi\cap A}(y) I_{|x-y|<R}(x,y)}{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} I_A(s) I_A(t) I_{\{|s-t|<R\}}(s,t)dsdt}dxdy. \tag{6}$$

for which the numerator of its integrand is exactly that found in (3). With a change of variables, this can be rewritten as

$$\int_{u \in S_d(R)}\int_{\mathbb{R}^d} \frac{I_{\Phi\cap A}(x) I_{\Phi\cap A}(x+u)}{\int_{S_d(R)} \nu_d(A \cap A_v)dv}dxdu.$$

The numerator of the integrand can be rewritten as the sum of the numerically identical but formally distinct functions

$$\phi_1(x) = I_{\Phi\cap A}(x) I_{\Phi\cap A}(x+u)$$

and

$$\phi_2(x) = I_{\Phi\cap A}(x-u) I_{\Phi\cap A}(x)$$

with the two forms reflecting the symmetry of the original numerator in (6). The equivalent version of the conditional expectation in (4) is thus

$$\begin{aligned} & E[\phi_1(x) \mid x \in \Phi \cap A] + E[\phi_2(x) \mid x \in \Phi \cap A] \\ & \quad = \Pr[x+u \in \Phi | x \in \Phi] I_{\Phi\cap A\cap A_u}(x) + \Pr[x-u \in \Phi | x \in \Phi] I_{\Phi\cap A\cap A_{-u}}(x) \\ & \quad = \Pr[x+u \in \Phi | x \in \Phi] \left(I_{\Phi\cap A\cap A_u}(x) + I_{\Phi\cap A\cap A_{-u}}(x) \right). \end{aligned} \tag{7}$$

In (4), the finite number of observations involved are required to be independent. For random sets, the analogous condition would be that for points separated by a distance $|u|$, the event $x \in \Phi$ is independent of the event $x+u \in \Phi$. This suggests that the conditional probability in (7) should be replaced by m_1. Replacing the sum in (4) with integrals over all relevant values of x and u in (7), the analogue of $\sum_{j=1}^{n} g(x_j)$ is

$$m_1 \left(\widehat{m}_1^0[\check{0}, r] + \widehat{m}_1^0[0, \check{r}] \right) \tag{8}$$

where

$$\widehat{m}_1^0[\check{0}, r] = \frac{\int_{u\in S_d(R)} \nu_d(\Phi \cap A \cap A_u) du}{\int_{s\in S_d(R)} \nu_d(A \cap A_s) ds} \tag{9}$$

$$\widehat{m}_1^0[0, \check{r}] = \frac{\int_{u\in S_d(R)} \nu_d(\Phi_u \cap A \cap A_{-u}) du}{\int_{s\in S_d(R)} \nu_d(A \cap A_s) ds}. \tag{10}$$

The analogue of the correction is found by subtracting from (8) a quantity that will give the bracketed term an expectation close to 0 and by replacing the parameters in this expression by their estimates, to yield:

$$\widehat{m}_1 \left(\widehat{m}_1^0 [\check{0}, r] + \widehat{m}_1^0 [0, \check{r}] - 2\widehat{m}_1 \right). \tag{11}$$

Given this, it suggests that the appropriate modification for the isotropic estimator $\widehat{m}_2^I(r)$ would be

$$\widehat{H}^I = \widehat{m}_1 \left(\widehat{m}_1^I [\check{0}, r] + \widehat{m}_1^I [0, \check{r}] - 2\widehat{m}_1 \right)$$

where

$$\widehat{m}_1^I\left[\check{0},r\right] = \frac{\int_{u\in C_d(R)} \nu_d(\Phi \cap A \cap A_u)du}{\int_{s\in C_d(R)} \nu_d(A \cap A_s)ds}$$
$$\widehat{m}_1^I\left[0,\check{r}\right] = \frac{\int_{u\in C_d(R)} \nu_d(\Phi_u \cap A \cap A_u)du}{\int_{s\in C_d(R)} \nu_d(A \cap A_s)ds}.$$

and $C_d(R)$ is the surface of the sphere in $\mathbb{R}^d$ of radius R. In the case of a non-isotropic random set process, choosing a single direction and length for r would suggest that the modification would be

$$\widehat{H} = \widehat{m}_1\left(\widehat{m}_1\left[\check{0},r\right] + \widehat{m}_1\left[0,\check{r}\right] - 2\widehat{m_1}\right)$$

where

$$\widehat{m}_1\left[\check{0},r\right] = \frac{\nu_d(\Phi \cap A \cap A_r)}{\nu_d(A \cap A_r)}$$
$$\widehat{m}_1\left[0,\check{r}\right] = \frac{\nu_d(\Phi_r \cap A \cap A_r)}{\nu_d(A \cap A_r)}.$$

Since $\kappa_2(r)$ and $\Sigma_2(r)$ have first order Taylor series expansions that are equivalent, it is reasonable to conjecture that the use of the additive modifications would also improve the performance of the ordinary cumulants as well.

Given these forms of the additive modification, the modified second cumulant and second spatial cumulant estimators can be defined as

$$\widehat{\kappa}_2^I(R) = \widehat{m}_2^I(R) - \widehat{m}_1^2 \qquad \widehat{\Sigma}_2^I(R) = \frac{\widehat{m}_2^I(R)}{\widehat{m}_1^2}$$

when the isotropic modification is used,

$$\widehat{\kappa}_2^{I*}(R) = \widehat{m}_2^I(R) - \widehat{H}^I - \widehat{m}_1^2 \qquad \widehat{\Sigma}_2^{I*}(R) = \frac{\widehat{m}_2^I(R) - \widehat{H}^I}{\widehat{m}_1^2}$$

when the isotropic modification and additive modification are used, and

$$\widehat{\kappa}_2^*(R) = \widehat{m}_2(R) - \widehat{H} - \widehat{m}_1^2 \qquad \widehat{\Sigma}_2^*(R) = \frac{\widehat{m}_2(R) - \widehat{H}}{\widehat{m}_1^2}$$

when an additive modification is used alone. For the estimates of the correlation, similar estimators can be constructed by use of an appropriately modified second moment estimator.

4.2.2 The Intrinsic Modification And The Idea Of Balance

To interpret the action of the additive modification, it is useful to partition a window A into three disjoint sets:

$$A_0 = A \cap A_r^c \cap A_{-r}^c$$
$$A_1 = \left(A \cap A_r \cap A_{-r}^c\right) \cup \left(A \cap A_r^c \cap A_{-r}\right)$$
$$A_2 = A \cap A_r \cap A_{-r}.$$

On A_0, no points in the window are considered in the estimation of the second moment. Each point in A_2 is used twice in the estimation of $m_2(r)$, once as a member of $A \cap A_{-r}$ and once as a member of $A \cap A_r$, while the points in A_1 are only used in one of these two roles. By restating $\widehat{H}$ as

$$\widehat{H} = \widehat{m}_1 \left(\widehat{h}_0 + \widehat{h}_1 + \widehat{h}_2\right)$$

where

$$\widehat{h}_2 = \left(\frac{\nu_d(A_2)\left(2\nu_d(A_0) + \nu_d(A_1)\right)}{\nu_d(A \cap A_r)\,\nu_d(A)}\right) \frac{\nu_d(\Phi \cap A_2)}{\nu_d(A_2)}$$
$$\widehat{h}_1 = \left(\frac{\nu_d(A_1)\left(\nu_d(A_0) - \nu_d(A_2)\right)}{\nu_d(A \cap A_r)\,\nu_d(A)}\right) \frac{\nu_d(\Phi \cap A_1)}{\nu_d(A_1)}$$
$$\widehat{h}_0 = -\left(\frac{\nu_d(A_0)}{\nu_d(A)}\right) \frac{\nu_d(\Phi \cap A_0)}{\nu_d(A_0)}$$

This suggests that the modification works by reweighting $\widehat{m}_2(r)$ by means of first moment estimates on a partition of A whose form is determined by the manner in which the points in A are used in the estimation of $m_2(r)$. By assuming that Φ is on $\mathbb{R}$ and that A has length 1, it follows that $\widehat{H} = 0$ not only when $r = 0$ but also when $r = 0.5$, which correspond respectively to $A = A_2$ and $A = A_1$. Thus, if the modification has no effect when every point in the window is used in the estimate the second moment in the same way, then this suggests that it is correcting the imbalance in the way that points in A are used to estimate $m_2(r)$. In the sequel, $\widehat{\kappa}_2^*(r)$ and $\widehat{\Sigma}_2^*(r)$ will be referred to as *additively balanced* estimators.

If the modification $\widehat{H}$ is correcting for some sort of imbalance, then this suggests a second strategy for modifying the estimates. Instead of adding a term to modify the estimator, one could estimate the first moment only on the parts of the window used to estimate the second moment by means of

$$\widehat{\kappa}_2^{\circledast} = \widehat{m}_2(r) - \widehat{m}_1[\check{0}, r]\widehat{m}_1[0, \check{r}]$$
$$\text{and} \quad \widehat{\Sigma}_2^{\circledast} = \frac{\widehat{m}_2(r)}{\widehat{m}_1[\check{0}, r]\widehat{m}_1[0, \check{r}]}.$$

Such estimates will be referred to as *intrinsically balanced* estimators. Intrinsic balancing can be much more easily extended to higher order spatial cumulants additive balancing. It may also be combined with the isotropic modifications as follows:

$$\widehat{\kappa}_2^{I\circledast} = \widehat{m}_2^I(r) - \widehat{m}_1^I[\check{0}, r]\widehat{m}_1^I[0, \check{r}]$$
$$\widehat{\Sigma}_2^{I\circledast} = \frac{\widehat{m}_2^I(r)}{\widehat{m}_1^I[\check{0}, r]\widehat{m}_1^I[0, \check{r}]}.$$

5 Asymptotic Analysis of Estimator Variance

In order to determine the effect of these modifications, it is necessary to make use of asymptotic methods. Analytical expressions have not been found for the moments of most random set models, and even for the simplest Boolean models these functions can often be extremely difficult to use. Again, the most intuitive asymptotic regime produces little improvement, but clearly interpretable asymptotic results can be found through use of a regime that accommodates balancing.

5.1 Asymptotic Regimes and Assumptions

The most straightforward method of increasing the information available is to increase the size of the window, and then examine the bias or variance of the estimator as the size becomes infinite. This method, which could be referred to as *expanding window asymptotics*, is not well suited to any estimator that involves a balancing effect. If the window A is partitioned into the sets A_0, A_1, and A_2, then as the window expands A_2 comes to dominate A and the effects of balancing are pushed out of the leading term of any asymptotic result. In addition, these non-leading terms are often extremely complicated and intractable integrals, and this makes it difficult to assess if the asymptotic results indicate that the modifications improve or impair estimator performance.

In contrast, if Φ is considered to be scaled down in every dimension by some factor $1/\beta$ while the window A and the vector r used to define $m_2(r)$ is kept intact, then this will keep the relative sizes of the A_0, A_1 and A_2 constant as the Φ is reduced. This regime, called the *shrinking process asymptotics*, is based on a similar regime used by Stein [11] in the investigation of the modifications to the K-function in point process theory. It is equivalent to letting the length of r expand in proportion to the increase in A, and if the dependence structure of Φ does fade away at great distances, then this ensures that the estimate of $\kappa_2(r)$ or $\Sigma_2(r)$ is being undertaken under conditions where the events $x \in \Phi$ and $x + r \in \Phi$ are essentially independent. Since estimating the region where $\kappa_2(r)$ or $\Sigma_2(r)$ take on constant values is one of the primary uses of these

functions, the asymptotic bias and variance under this regime at great distances may provide a hint as to how these function estimates behave at shorter distances.

The notion of the dependence structure breaking down over a large distances can also be formulated more precisely for higher order moments by means of the following condition, which is most clearly stated in terms of the ordinary moments of Φ:

Mixing Condition 5.1. Let Φ be a stationary random set process, defined on $\mathbb{R}^d$. Then for any $k \in \mathbb{N}^{>0}$, any $x_1, \ldots, x_k \in \mathbb{R}^d$ and any l such that $1 \leq l \leq k$, if the minimum distance between the points in $\{x_1, \ldots, x_l\}$ and in $\{x_{l+1}, \ldots, x_k\}$ is δ, then

$$|M_k(x_1, \ldots, x_k) - M_l(x_1, \ldots, x_l) M_{k-l}(x_{l+1}, \ldots, x_k)| = o(\delta^{-(d+1)})$$

Such a condition was first used by Mase to establish bounds on the fourth cumulant of a point process [13].

5.2 Effects of the Modifications

Through use of the shrinking process asymptotics and the mixing condition, an easily interpretable result concerning the effectiveness of the modifications can be found.

Proposition 5.1. *Let Φ be a stationary random set process, examined using the SP asymptotic regime. Assume that the Mixing Condition holds for the 2nd through 4th moments. The asymptotic variance of the unmodified estimator $\widehat{\kappa}_2(r)$ for any $r \in \mathbb{R}^d$ is then*

$$\frac{1}{\beta^d} \left\{ \frac{1}{\nu_d(A \cap A_r)} \int_{\mathbb{R}^d} \kappa_2(x)^2 dx + 2Q_0 \int_{\mathbb{R}^d} \kappa_1^2 \kappa_2(x) dx \right\} + O\left(\beta^{-(d+1)}\right) \tag{12}$$

where

$$Q_0 = \frac{\nu_d(A \cap A_r) + \nu_d(A \cap A_r \cap A_{-r})}{\nu_d(A \cap A_r)^2} - \frac{2}{\nu_d(A)} \geq 0. \tag{13}$$

With the balance modifications, the asymptotic variances of both $\widehat{\kappa}_2^*(r)$ and $\widehat{\kappa}_2^{\circledast}(r)$ are

$$\frac{1}{\beta^d} \left\{ \frac{1}{\nu_d(A \cap A_r)} \int_{\mathbb{R}^d} \kappa_2(x)^2 dx \right\} + O\left(\beta^{-(d+1)}\right).$$

The asymptotic variance will be reduced by the balance modification, save in the case where A consists of two components of identical shape, size, and orientation separated by the vector r; in that case there will be no improvement. With the isotropic modification, the asymptotic variance of $\widehat{\kappa}_2^I(R)$ is

$$\frac{4}{\beta^d} \left(Q_I - \frac{1}{\nu_d(A)} \right) \int_{\mathbb{R}^d} \kappa_1^2 \kappa_2(x) dx + O\left(\beta^{-(d+1)}\right) \tag{14}$$

where

$$Q_I = \frac{\int_{C_d(R)} \nu_d(A \cap A_{r_1} \cap A_{r_2}) dr_1 dr_2}{\left(\int_{C_d(R)} \nu_d(A \cap A_r) dr\right)^2} \tag{15}$$

For the balanced and isotropic estimators $\widehat{\kappa}_2^{I*}(r)$ *and* $\widehat{\kappa}_2^{I\circledast}(r)$ *the asymptotic variance is* $o(\beta^{-d})$.

Proof of Proposition 5.1

The variance of the second cumulant estimator $\widehat{\kappa}_2(r)$ will be

$$\left(\mathrm{E}\left[\widehat{m}_2(r)^2\right] - m_2(r)^2\right) + \left(\mathrm{E}\left[\widehat{m}_2(r)\widehat{m}_1^2\right] - m_2(r)\,\mathrm{E}\left[\widehat{m}_1^2\right]\right) + \left(\mathrm{E}\left[\widehat{m}_1^4\right] - \mathrm{E}\left[\widehat{m}_1^2\right]^2\right) \tag{16}$$

but the $\mathrm{E}\left[\widehat{m}_1^2\right]$ terms will cancel out in the SP asymptotics and these can be regarded as equivalent to m_1^2 in the context of that regime. Each of these differences can be rewritten as an integral and each of these integrals will be finite if the mixing condition is true. When the shrinking process asymptotics are implemented, the multiplier β is applied to each argument of the moments in the integrands of these integrals:

$$\int_{\mathbb{R}^d} \left(m_4(-\beta r, \beta u, \beta u - \beta r) - m_2(\beta r)^2\right) \frac{\nu_d\left(A \cap A_r \cap A_{-u} \cap A_{-u+r}\right)}{\nu_d\left(A \cap A_r\right)^2} du \tag{17}$$

$$- 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(m_4(-\beta r, \beta u_1, \beta u_2) - m_2(\beta r) m_1^2\right) \frac{\nu_d\left(A \cap A_r \cap A_{-u_1} \cap A_{-u_2}\right)}{\nu_d\left(A \cap A_r\right)\nu_d\left(A\right)^2} du_1 du_2 \tag{18}$$

$$+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(m_4(\beta u_1, \beta u_2, \beta u_3) - m_1^4\right) \frac{\nu_d\left(A \cap A_{-u_1} \cap A_{-u_2} \cap A_{-u_3}\right)}{\nu_d\left(A\right)^4} du_1 du_2 du_3. \tag{19}$$

For each of these integrals, an $O(\beta^{-d})$ term can occur only when a pair of arguments in the fourth moments in the integrands are constrained to be close. If more than two arguments are constrained to be close, the resulting terms will be $O(\beta^{-2d})$ or $O(\beta^{-3d})$ and so such interactions will not affect the leading term. By considering each such interaction and by making use of the Mixing Condition and the Dominated Convergence Theorem, the integral (17) has an asymptotic expansion with leading term

$$\frac{1}{\beta^d}\left(\frac{1}{\nu_d\left(A \cap A_r\right)}\int_{\mathbb{R}^d} \kappa_2(u)^2 du + 2\frac{\nu_d\left(A \cap A_r\right) + \nu_d\left(A \cap A_r \cap A_{-r}\right)}{\nu_d\left(A \cap A_r\right)^2}\int_{\mathbb{R}^d} \kappa_1^2 \kappa_2(u) du\right).$$

Combining this with the $O(\beta^{-d})$ terms from the integrals (18) and (19), the asymptotic variance (12) is obtained. The results for the balanced estimator are found by similar arguments.

When the isotropic modification is considered, the second moment estimators in (16) are replaced by their isotropic versions, and the integral (17) is replaced by

$$\int_{C_d(R)}\int_{C_d(R)}\int_{\mathbb{R}^d} \left(m_4(-\beta r_1, \beta u, \beta u - \beta r_2) - m_2(\beta r_1) m_2(\beta r_2)\right) \times \frac{\nu_d\left(A \cap A_{r_1} \cap A_{-u} \cap A_{-u+r_2}\right)}{\left(\int_{C_d(R)} \nu_d\left(A \cap A_s\right) ds\right)^2} du dr_1 dr_2$$

where $C_d(R)$ is the surface of the sphere of radius R centered at 0 in $\mathbb{R}^d$. In this case, the arguments of the fourth moment are not constrained to form a parallelogram as they were in (17), and so the squared second cumulant integral does not appear in the leading term of its asymptotic expansion. Instead, the leading term is

$$\frac{4}{\beta^d}\int_{C_d(R)}\int_{C_d(R)}\frac{\nu_d\left(A\cap A_{r_1}\cap A_{r_2}\right)}{\nu_d\left(A\cap A_{r_1}\right)\nu_d\left(A\cap A_{r_2}\right)}dr_1dr_2\int_{\mathbb{R}^d}\kappa_1^2\kappa_2(u)du.$$

The other two integrals produce leading terms identical to those in the non-isotropic case, and so when put together yield (14). The results for the balanced isotropic estimator are found by similar arguments. ■

Similar results can be found for the modified estimates of $\Sigma_2(r)$ and $\rho(r)$, but these results are purely formal. They are based upon Taylor series expansions of the ratios involved in these statistics, but no bounds can be placed on the size of the remainder term in these expressions since tractable analytic expressions for the moments are not available.

The modifications also have a distinct effects on the bias of the estimates.

Proposition 5.2. *Let Φ be a stationary random set process, examined using the SP asymptotic regime. Assume that the Mixing Condition holds for the 2nd through 4th moments. Then the leading term of the asymptotic bias of the second cumulant has the form*

$$\frac{K}{\beta^d}\int_{\mathbb{R}^d}\kappa_2(t)dt$$

where

$$K=-\frac{\nu_d\left(A\cap A_r\cap A_{-r}\right)}{\nu_d\left(A\cap A_r\right)^2} \qquad \text{for} \quad \widehat{\kappa}_2^{\circledast}(r)$$

$$K=-Q_I \qquad \text{for} \quad \widehat{\kappa}_2^{I\circledast}(r)$$

$$K=-\frac{1}{\nu_d\left(A\right)} \qquad \text{for} \quad \widehat{\kappa}_2(r),\widehat{\kappa}_2^{*}(r),\widehat{\kappa}_2^{I}(r), \quad \text{and} \quad \widehat{\kappa}_2^{I*}(r)$$

Proof of Proposition 5.2

In the case of the second cumulant function, it is straightforward to show that

$$\begin{aligned}
\mathrm{E}[\widehat{\kappa}_2(r) - \kappa_2(r)] &= -\operatorname{var}[\widehat{m}_1] \\
\mathrm{E}[\widehat{\kappa}_2^*(r) - \kappa_2(r)] &= \operatorname{var}[\widehat{m}_1] - \operatorname{cov}[\widehat{m}_1, \widehat{m}_1[\check{0}, r]] - \operatorname{cov}[\widehat{m}_1, \widehat{m}_1[0, \check{r}]] \\
\mathrm{E}[\widehat{\kappa}_2^{\circledast}(r) - \kappa_2(r)] &= -\operatorname{cov}[\widehat{m}_1[\check{0}, r], \widehat{m}_1[0, \check{r}]] \\
\mathrm{E}[\widehat{\kappa}_2^I(r) - \kappa_2(r)] &= -\operatorname{var}[\widehat{m}_1] \\
\mathrm{E}[\widehat{\kappa}_2^{I*}(r) - \kappa_2(r)] &= \operatorname{var}[\widehat{m}_1] - \operatorname{cov}[\widehat{m}_1, \widehat{m}_1^I[\check{0}, r]] - \operatorname{cov}[\widehat{m}_1, \widehat{m}_1^I[0, \check{r}]] \\
\mathrm{E}[\widehat{\kappa}_2^{I\circledast}(r) - \kappa_2(r)] &= -\operatorname{cov}[\widehat{m}_1^I[\check{0}, r], \widehat{m}_1^I[0, \check{r}]].
\end{aligned} \tag{20}$$

From asymptotic expansions of these variances and covariances, the leading terms given above follow. ■

6 Simulations

In order to establish that the asymptotic results of Propositions 5.1 and 5.2 are actually indicating an effect of useful size, it is necessary to determine how modified estimators perform on simulated data. To this end, realizations of a simple Boolean model were used. These realizations were generated to fill a 10 unit by 10 unit window, with grains that were circles of $\mu = 1$ unit in diameter.

Estimation of the moment was undertaken by an algorithm that was designed to make accurate estimates of moments from each realization [14]. Each of the 50 realizations generated was discretized into an array of thin strips which were then divided into covered and uncovered regions. By using carefully chosen orientations, it was possible to accurately calculate $\widehat{m}^2(r)$ for 16 different directions of r while using only two discretizations of the realization, thus ensuring that the majority of the estimated standard deviation came from the process, rather than from the determination of the estimate.

7 Discussion

So far, a result for point processes has suggested a modification that might reduce the variance of estimators of $\Sigma_2(r)$. This in turn suggested a second form of the modification based on considerations specific to random sets, and both of those modifications could be modified to act on estimators of $\kappa_2(r)$. For the modified estimators of $\kappa_2(r)$, rigorous asymptotic results were found that suggested that the modifications would have a beneficial effect on estimator variance, and formal results suggested that these results would carry over to estimators of $\Sigma_2(r)$. Having accumulated all of these suggested results, the simulations indicate

that the modifications do have a significant and understandable effect on estimator variance for estimators of both functions.

7.1 Effects on Estimator Variation

The effect of the balance and isotropic effects are best seen when considering estimates of $\kappa_2(|r|)$ for a single realization. Two randomly chosen examples of estimates of this function are provided for the Boolean model with $m_1 = 0.1$ in Figure 1, illustrating how both modifications draw the estimated function in closer to the line $\kappa_2(|r|) = 0$ when $R > \mu$. Since identification of this portion of $\kappa_2(|r|)$ is essential to being able to establish the limits of the dependence structure, the improvements that the modifications can induce are of potentially great use in the analysis of single samples. In excess of thirty realizations were examined for this particular model, and in every case markedly improved estimates of $\kappa_2(|r|) = 0$ were found. For $m_1 = 0.1$ the isotropic modification had the dominant corrective effect but this dominance was reversed when $m_1 = 0.9$. The modifications had no obvious beneficial or detrimental effect when the dependence structure was present.

The second use of $\kappa_2(|r|)$ involves the detection of repulsion between the grains or, equivalently, between the underlying germs. Since the window was relatively small, it was possible that the local behaviour of the Boolean model could exhibit some repulsive behaviour purely by chance, especially when m_1 was small. This was often observed and is present in the second plot of Figure 1, but the modifications were found to make this worse as often as often as they were found to reduce it. It is conjectured that this type of behaviour could only be removed through use of larger windows, rather than through the modifications suggested here.

When the average behaviour of modified estimates is considered, the improvements brought about by the balancing and isotropic modifications are equally dramatic. As can be seen in Figures 2-4, the degree of sampling variance reduction depends upon both m_1 and the between r and the size of $A \cap A_r$, both of which are indications of the amount of information available for standardized moment estimation. The effects of both factors must be considered separately on the regions where $|r| < \mu$ and $|r| > \mu$ for this sort of Boolean model, as the dependence structure fades away entirely when $|r| = \mu$.

On the region where the dependence structure does exist, the balance modification has very little effect on estimator variation for small $|r|$. This is not surprising, as the heuristic construction of the balance modification required an absence of any dependence structure. The isotropic modification also has little effect at short distances since an alteration in direction of $|r|$ does not radically change that which is being compared. As $|r|$ increases, the event that the points $x \in \Phi$ and $x + r \in \Phi$ is more likely to involve two

points that lie in distinct grains, and so the modifications begin to have an effect.

On the region where the dependence structure has vanished, the modifications have significant and explicable effects. Here, $\widehat{\kappa}_2(r)$ becomes a comparison between two different methods of estimating m_1^2 in which different amounts of information are used for the two estimates. As the value of $|r|$ increases, estimator variance increases as less information is available for estimating $m_2(r)$, but this increase is not uniform. While r in these examples is not parallel to the side of a window, there is a slight decrease in the variance as $|r|$ approaches half of the maximum distance that can be used in the estimation of $m_2(r)$ in the direction of r. At that point, $\nu_d(A_1)$ is maximized for all distances at which $\nu_d(A_2) = 0$, which suggests that the elimination of A_2 produces a beneficial effect. This effect occurs at different distances depending on the orientation of r, and so the dip in the variance of the unbalanced isotropic estimator occurs at a greater value of R. No such dip occurs in the balanced estimates, where compensation for imbalances has already occurred.

The isotropic modification reduces estimator variance by making use of more available information. When m_1 is small, this effect is relatively large, as it allows many more comparisons of $\widehat{m}_2(r)$ and $\widehat{m}_1^2$ to be made, since the grains of Φ are sparse. When they are plentiful, as when $m_1 = 0.9$, the effect is much less significant until the value of $|r|$ becomes large. Under those conditions, the increased size of A_1 for vectors r whose orientation is close to the diagonal of the window produces an increased degree of variance reduction.

The balance modification, at least in the intrinsic case, works by forcing the estimates of m_1 used in estimating $\kappa_2(r)$ to be determined only on those parts of the window used to estimate $m_2(r)$. These first moments estimates have greater variance than $\widehat{m}_1$, but they are much more correlated to $\widehat{m}_2(r)$ than $\widehat{m}_1$ is, particularly when $|r|$ is large. The extra information available in $\widehat{m}_1$ lends nothing to the comparison of $\widehat{m}_2(r)$ and $\widehat{m}_1^2$ on A_1, and so contributes only excess variation. In each of the three cases considered, the balanced estimators had noticeably less variance than the unmodified estimators. The degree of variance reduction increased as m_1 increased, on account of the larger number of comparisons of estimators of $m_2(r)$ and m_1^2 that occurred. It must be noted that while $SD[\widehat{\kappa}_2^{I\circledast}(r)]$ variation in Figure 4 seems almost constant for the balanced and balanced isotropic estimators, in all cases this is almost a rescaling of the dependence of the estimate of $SD[\widehat{\kappa}_2(r)]$ on $|r|$. The simulations showed no real difference between intrinsically and additively balanced estimates, both of which have the same SP asymptotic behaviour in spite of their different functional forms.

When both modifications are used, the estimated standard deviation of the cumulant estimates is further reduced, providing the greatest benefit in all cases examined. There appears to be no deleterious effect of using both together, although the combined effect when m_1 is small may not be so different from when the

isotropic modification is used alone.

Although it was not possible to prove a rigorous asymptotic result for the bias and variance of $\Sigma_2(r)$, simulation results showed a very similar dependence of estimator variance on r and m_1. The only major difference occurred when $m_1 = 0.1$, as shown in Figure 5, and this related to the different effects of intrinsic and additive balancing. When $|r|$ was relatively large, very small values of $\widehat{m}_1[\check{0}, r]$ or $\widehat{m}_1[0, \check{r}]$ can increase the variance of the intrinsically balanced estimator through inflating the value of the ratio estimate. Given that the spatial cumulant $\Sigma_2(r)$ may belong to class of dependence measures better suited than ordinary cumulants to the description of random sets [9], these modifications will prove useful in any simulation-based studies of their behaviour.

Examination of certain covariances between estimators lends support to the explanations given for the improvements caused by the balance modifications. If covariances between estimators of $\kappa_2(5)$ and $\kappa_2(5+s)$ are considered as in Figure 6, then the use of balance modifications decreases the correlation between other estimates as $|s|$ increases. For the unbalanced estimators, the same information would be used to estimate m_1 regardless of the value of s, and so $\mathrm{cov}[\widehat{\kappa}_2(r), \widehat{\kappa}_2(r+s)]$ would be higher than for balanced estimators. If covariances between estimators of $\kappa_2(r)$ and $\widehat{m}_1$ are considered, Figure 7 shows that the use of balance modifications increases the degree of correlation between these estimators. In a Boolean model, the dependence of $m_2(r)$ on the intensity of the underlying Poisson germ process is not removed by either of the two standardizations. In the case of this particular Boolean model,

$$\kappa_2(r) = (1-m_1)^2 \left(m_1^{-\frac{4\psi(|r|)}{\pi\mu^2}} - 1 \right)$$

where $\psi(|r|)$ is the area of the intersection of two discs of diameter μ whose centres are separated by a distance $|r|$. Any estimator of $\kappa_2(r)$ that includes information about m_1 that is not used in the estimation of $m_2(r)$ would then be expected to yield a poorer estimate of this function. This effect is greatly reduced for balanced estimates of $\Sigma_2(r)$ as shown by Figure 8, but this cannot be explained by any formal results involving Taylor series.

7.2 Effect of Modifications on Other Estimator Properties

The bias results indicate an additional subtle aspect to the notion of balancing. When the intrinsic modification is used alone, 20 indicates that the bias disappears entirely when $|r|$ becomes sufficiently large that the estimates on the two components of A_1 become independent. Under these conditions, both of the intrinsically balanced estimates will be unbiased. This will not be true for intrinsically balanced estimators

of $\Sigma_2(r)$, but it is expected that similar reductions in bias will be found. The bias in the simulations was so small as to be hidden by discretization errors in the calculation of the moments.

The distributions of the six estimators of $\kappa_2(r)$ were compared to normal distribution functions by means of Q-Q plots, but no systematic differences were found between modified and unmodified estimators. While there were occasional signs of non-normality, these were mostly in the form of tendencies towards skewness that followed no clear pattern. This is in marked contrast to the case of point processes, where Stein [12] demonstrated that the analogous modified K-function involved a degenerate U-statistic. As indicated by Hall [15], degenerate U-statistics often possess non-normal sampling distributions whose limiting laws are often very difficult to determine.

8 Summary and Conclusions

Given the theoretical and simulation results above, intrinsically balanced estimators of $\kappa_2(r)$ are the best ones to use with respect to minimizing estimator bias and variance. While not intuitive, they do make the best use of available information when estimating $\kappa_2(r)$, excluding extraneous information that would add nothing to the basic comparison implicit in the cumulant function. While the asymptotic results involving estimators of $\Sigma_2(r)$ are formal, the simulation results suggest that intrinsic balancing is also very effective for estimates of this parameter.

From the point of view of methodology, the arguments used here indicate a useful way in which ideas from point process theory can enable results to be found for general random set processes. An asymptotic method that is mathematically convenient for point processes becomes one that makes geometric sense for random sets, while a modification to a point process estimator that is based on the theory of U-statistics becomes one that produces a notion of balanced estimator, an idea which makes no sense at all in the point process context. Boolean models were not required in any of the theoretical results, and were used only as a method for producing simulation results that could be easily interpreted. In the further study of the otherwise mathematically intractable germ-grain models, these heuristic methods may prove to be effective at producing useful statistical techniques for engineering applications in situations where exact analytical results are impossible to obtain.

9 Acknowledgments

The problem that stimulated this research was proposed by Michael Stein, who also supervised the Ph.D. dissertation that emerged from it. Funding for this research was provided by the University of Chicago Louis Block Fund and NSF Grants DMS 89-05292 and DMS 92-08758.

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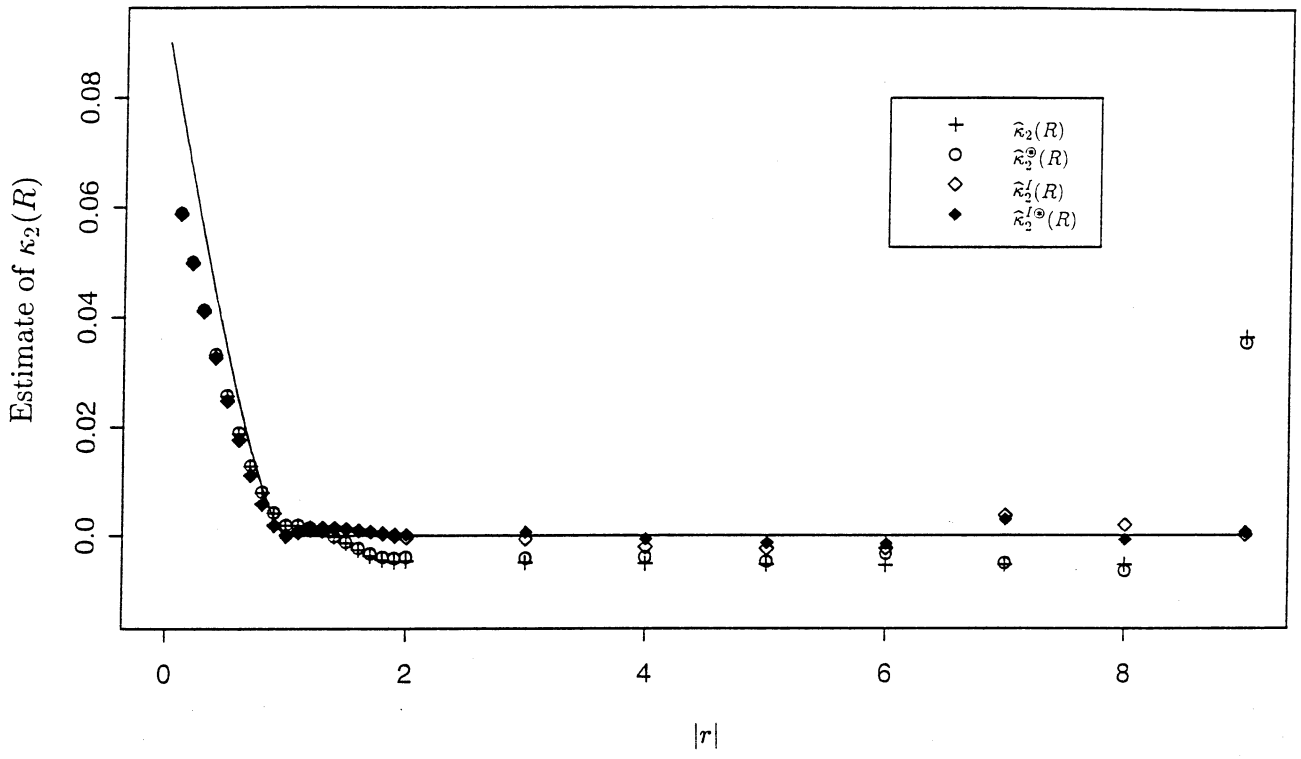

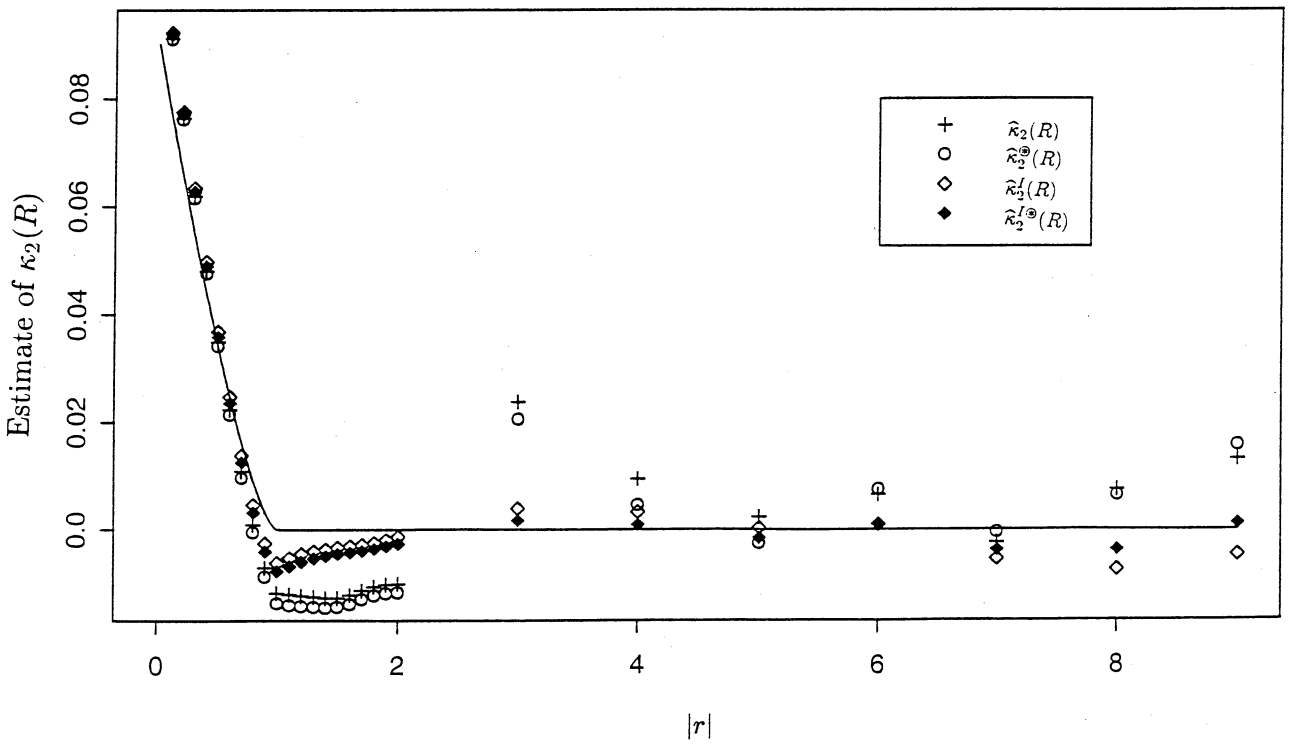

Figure 1: Estimates of $\kappa_2(R)$ for two simulations of the Boolean Model with $m_1 = 0.1$. Solid line is a graph of $\kappa_2(R)$.

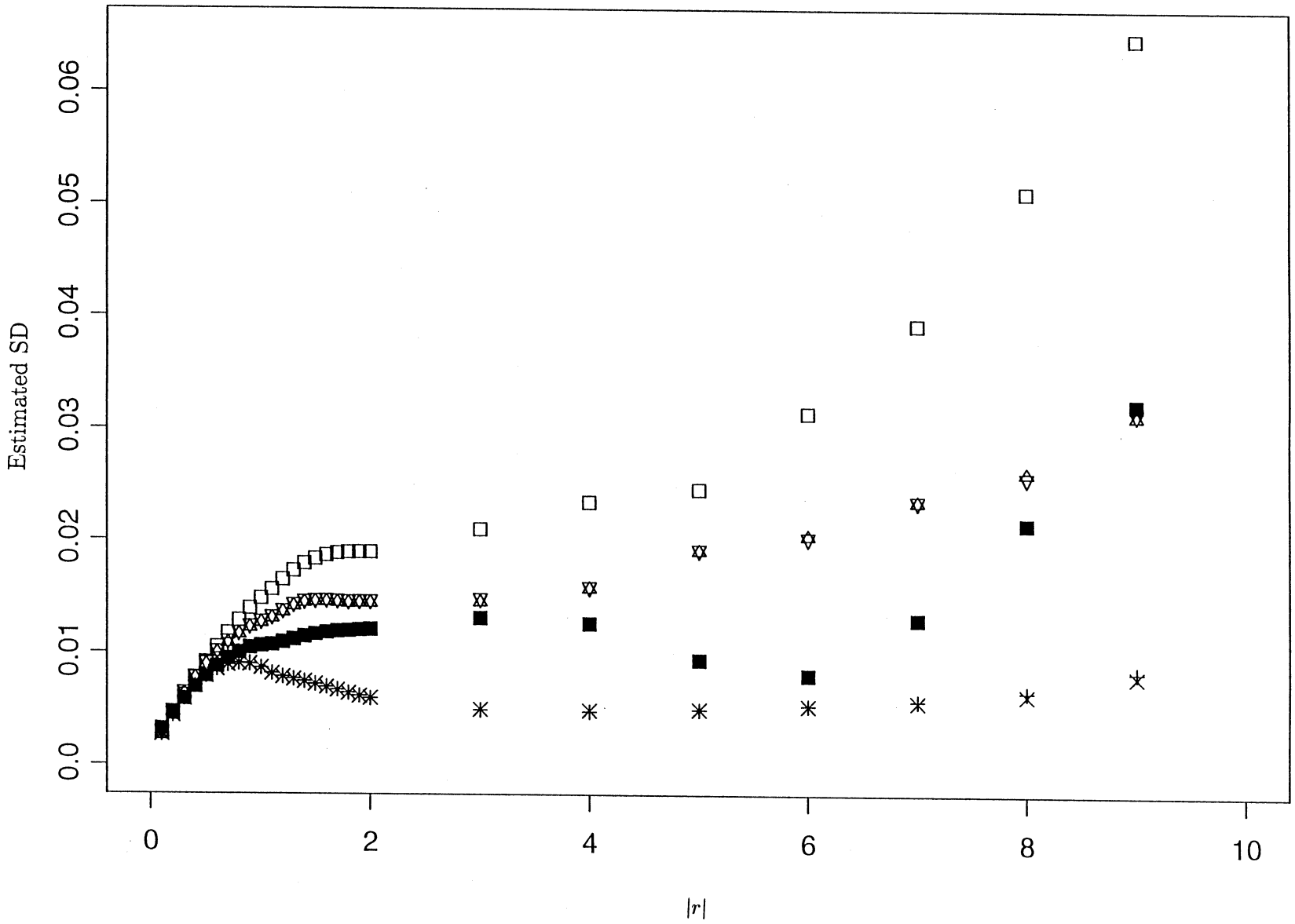

Figure 3:Estimated standard deviation of estimators of $\kappa_2(r)$ for the Boolean Model with $m_1 = 0.5$.

$\widehat{\kappa}_2(r)$ □; $\widehat{\kappa}_2^*(r)$ △; $\widehat{\kappa}_2^{\circledast}(r)$ ▽; $\widehat{\kappa}_2^I(r)$ ■; $\widehat{\kappa}_2^{I*}(r)$ +; $\widehat{\kappa}_2^{I\circledast}(r)$ ×

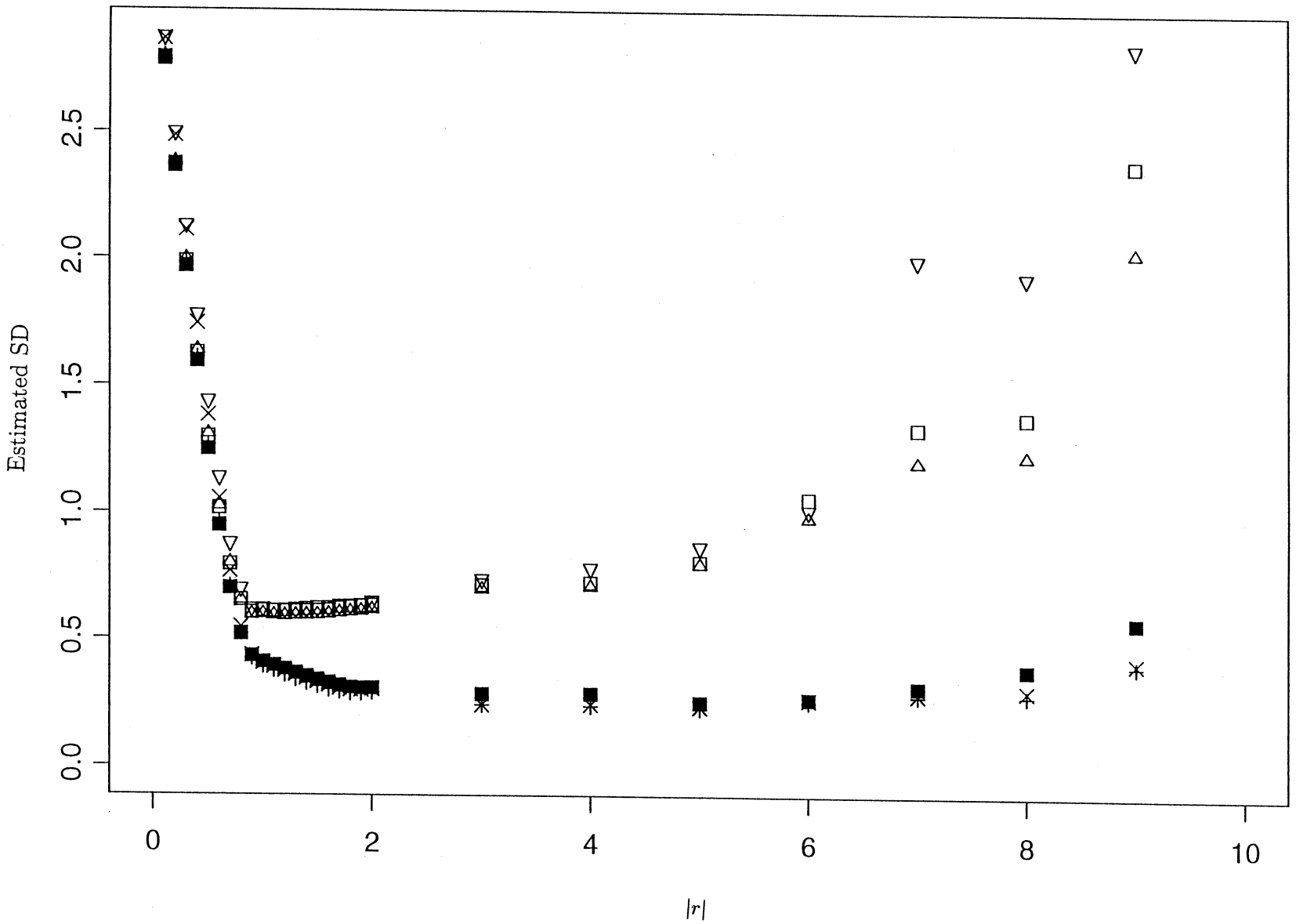

Figure 5: Estimated standard deviation of estimators of $\Sigma_2(r)$ for the Boolean Model with $m_1 = 0.1$.

$\widehat{\kappa}_2(r)$ □; $\widehat{\kappa}_2^*(r)$ △; $\widehat{\kappa}_2^{\circledast}(r)$ ▽; $\widehat{\kappa}_2^I(r)$ ■; $\widehat{\kappa}_2^{I*}(r)$ +; $\widehat{\kappa}_2^{I\circledast}(r)$ ×

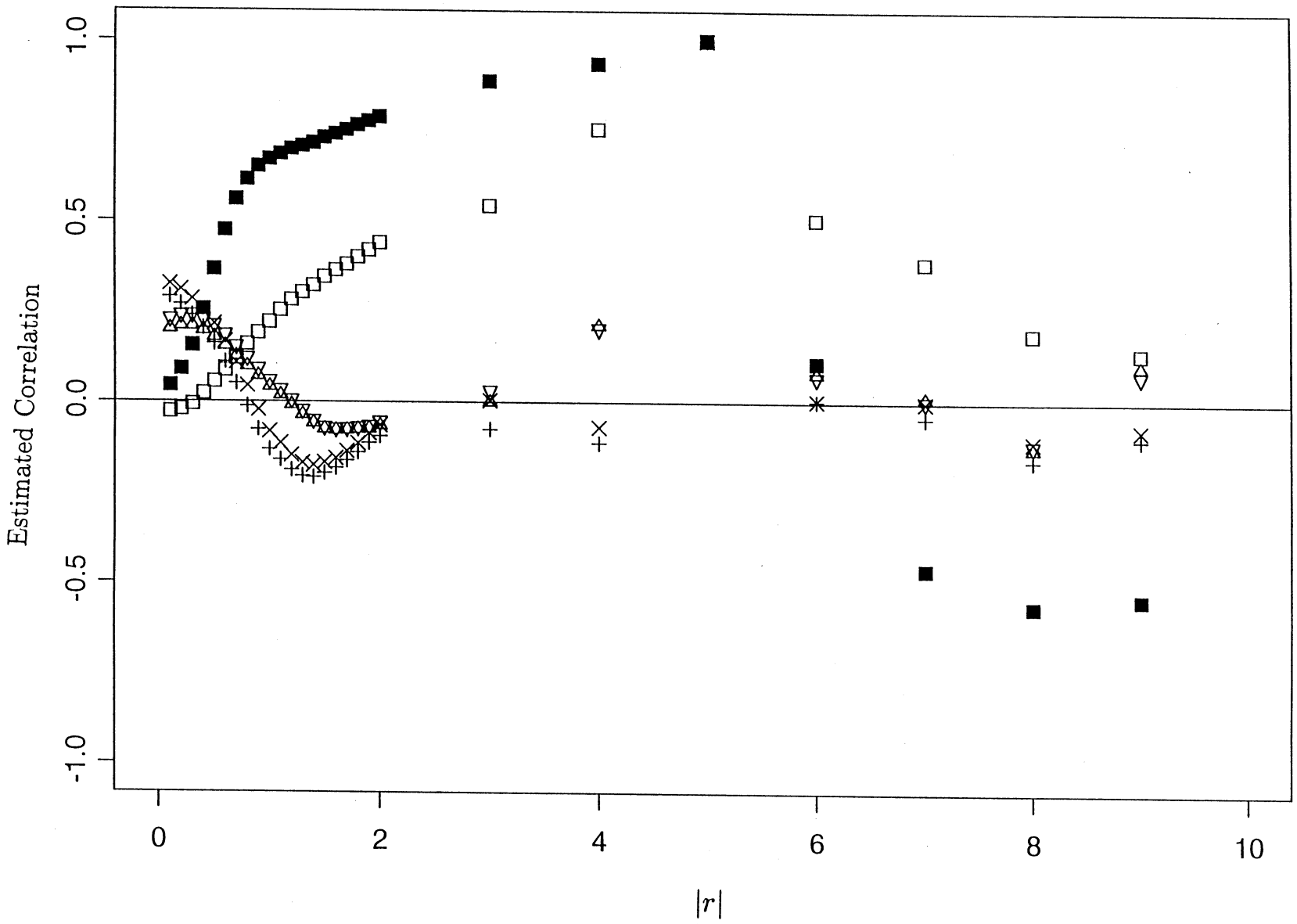

Figure 6: Estimated correlation of estimators of $\kappa_2(r)$ and $\kappa_2(5)$ for the Boolean Model with $m_1 = 0.9$.

$\widehat{\kappa}_2(r)$ □; $\widehat{\kappa}_2^*(r)$ △; $\widehat{\kappa}_2^{\circledast}(r)$ ▽; $\widehat{\kappa}_2^I(r)$ ■; $\widehat{\kappa}_2^{I*}(r)$ +; $\widehat{\kappa}_2^{I\circledast}(r)$ ×

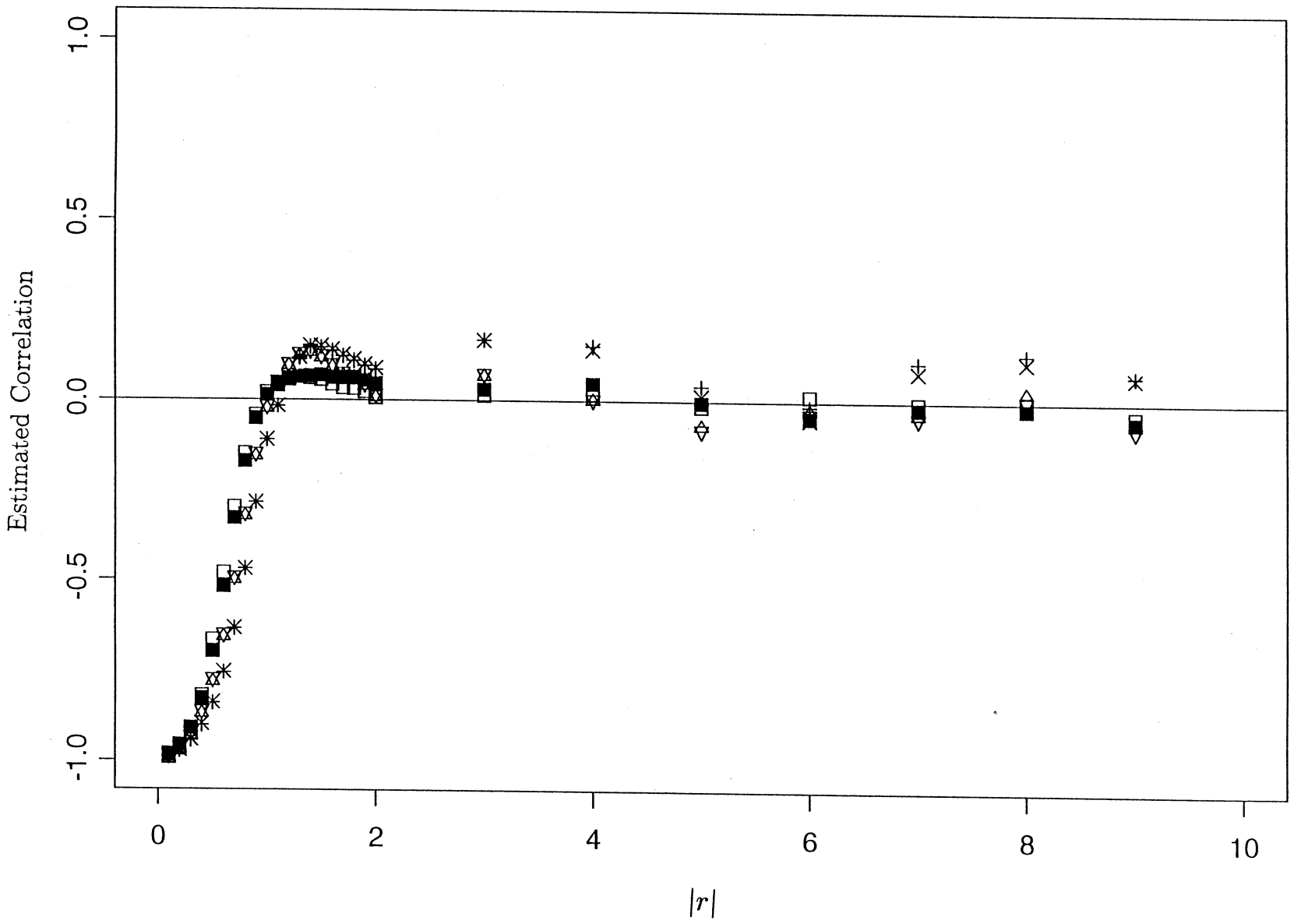

Figure 8: Estimated correlation of estimators of $\Sigma_2(r)$ and $\widehat{m}_1$ for the Boolean Model with $m_1 = 0.9$.

$\widehat{\kappa}_2(r)$ $\square$; $\widehat{\kappa}_2^*(r)$ $\triangle$; $\widehat{\kappa}_2^{\circledast}(r)$ $\triangledown$; $\widehat{\kappa}_2^I(r)$ $\blacksquare$; $\widehat{\kappa}_2^{I*}(r)$ $+$; $\widehat{\kappa}_2^{I\circledast}(r)$ $\times$